

Poisson geometry and Morita equivalence

Henrique Bursztyn*

Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3, Canada

Alan Weinstein †

Department of Mathematics
University of California, Berkeley
CA, 94720-3840 USA
and
Mathematical Sciences Research Institute
17 Gauss Way
Berkeley, CA 94720-5070 USA

Contents

1	Introduction	2
2	Poisson geometry and some generalizations	3
2.1	Poisson manifolds	3
2.2	Dirac structures	4
2.3	Twisted structures	8
2.4	Symplectic leaves and local structure of Poisson manifolds	9
2.5	Presymplectic leaves and Poisson quotients of Dirac manifolds	11
2.6	Poisson maps	13
2.7	Dirac maps	14
3	Algebraic Morita equivalence	17
3.1	Ring-theoretic Morita equivalence	17
3.2	Strong Morita equivalence of C^* -algebras	20
3.3	Morita equivalence of deformed algebras	22
4	Geometric Morita equivalence	25
4.1	Representations and tensor product	25
4.2	Symplectic groupoids	28
4.3	Morita equivalence for groups and groupoids	32

*henrique@math.toronto.edu, research partially supported by DAAD

†alanw@math.berkeley.edu, research partially supported by NSF grant DMS-0204100

MSC2000 Subject Classification Numbers: 53D17 (Primary), 58H05 16D90 (secondary)

Keywords: Picard group, Morita equivalence, Poisson manifold, symplectic groupoid, bimodule

4.4	Modules over Poisson manifolds and symplectic groupoid actions	34
4.5	Morita equivalence of Poisson manifolds and symplectic groupoids	36
4.6	Picard groups	40
4.7	Fibrating Poisson manifolds and Morita invariants	42
4.8	Gauge equivalence of Poisson structures	44
5	Geometric representation equivalence	46
5.1	Symplectic torsors	46
5.2	Symplectic categories	47
5.3	Symplectic categories of representations	48

1 Introduction

Poisson geometry is a “transitional” subject between noncommutative algebra and differential geometry (which could be seen as the study of a very special class of commutative algebras). The physical counterpart to this transition is the correspondence principle linking quantum to classical mechanics.

The main purpose of these notes is to present an aspect of Poisson geometry which is inherited from the noncommutative side: the notion of Morita equivalence, including the “self-equivalences” known as Picard groups.

In algebra, the importance of Morita equivalence lies in the fact that Morita equivalent algebras have, by definition, equivalent categories of modules. From this it follows that many other invariants, such as cohomology and deformation theory, are shared by all Morita equivalent algebras. In addition, one can sometimes understand the representation theory of a given algebra by analyzing that of a simpler representative of its Morita equivalence class. In Poisson geometry, the role of “modules” is played by Poisson maps from symplectic manifolds to a given Poisson manifold. The simplest such maps are the inclusions of symplectic leaves, and indeed the structure of the leaf space is a Morita invariant. (We will see that this leaf space sometimes has a more rigid structure than one might expect.)

The main theorem of algebraic Morita theory is that Morita equivalences are implemented by bimodules. The same thing turns out to be true in Poisson geometry, with the proper geometric definition of “bimodule”.

Here is a brief outline of what follows this introduction.

Section 2 is an introduction to Poisson geometry and some of its recent generalizations, including Dirac geometry and “twisted” Poisson geometry in the presence of a “background” closed 3-form. Both of these generalizations are used simultaneously to get a geometric understanding of new notions of symmetry of growing importance in mathematical physics, especially with background 3-forms arising throughout string theory (in the guise of the more familiar closed 2-forms on spaces of curves).

In Section 3, we review various flavors of the algebraic theory of Morita equivalence in a way which transfers easily to the geometric case. In fact, some of our examples come from geometry: algebras of smooth functions. Others come from the quantum side: operator algebras.

Section 4 is the heart of these notes, a presentation of the geometric Morita theory of Poisson manifolds and the closely related Morita theory of symplectic groupoids. We arrive at this theory via the Morita theory of Lie groupoids in general.

In Section 5, we attempt to remedy a defect in the theory of Chapter 4. Poisson manifolds with equivalent (even isomorphic) representation categories may not be Morita equivalent. We

introduce refined versions of the representation category (some of which are not really categories!) which do determine the Morita equivalence class. Much of the material in this section is new and has not yet appeared in print. (Some of it is based on discussions which came after the PQR Euroschool where this course was presented.)

Along the way, we comment on a pervasive problem in the geometric theory. Many constructions involve forming the leaf space of a foliation, but these leaf spaces are not always manifolds. We make some remarks about the use of differentiable stacks as a language for admitting pathological leaf spaces into the world of smooth geometry.

Acknowledgements:

We would like to thank all the organizers and participants at the Euroschool on Poisson Geometry, Deformation Quantization, and Representation Theory for the opportunity to present this short course, and for their feedback at the time of the School. We also thank Stefan Waldmann for his comments on the manuscript.

H.B. thanks Freiburg University for its hospitality while part of this work was being done.

2 Poisson geometry and some generalizations

2.1 Poisson manifolds

Let P be a smooth manifold. A **Poisson structure** on P is an \mathbb{R} -bilinear Lie bracket $\{\cdot, \cdot\}$ on $C^\infty(P)$ satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \quad \text{for all } f, g, h \in C^\infty(P). \quad (2.1)$$

A **Poisson algebra** is an associative algebra which is also a Lie algebra so that the associative multiplication and the Lie bracket are related by (2.1).

For a function $f \in C^\infty(P)$, the derivation $X_f = \{f, \cdot\}$ is called the **hamiltonian vector field** of f . If $X_f = 0$, we call f a **Casimir function** (see Remark 2.4). It follows from (2.1) that there exists a bivector field $\Pi \in \mathcal{X}^2(P) = \Gamma(\bigwedge^2 TP)$ such that

$$\{f, g\} = \Pi(df, dg);$$

the Jacobi identity for $\{\cdot, \cdot\}$ is equivalent to the condition $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket, see e.g. [84].

In local coordinates (x_1, \dots, x_n) , the tensor Π is determined by the matrix

$$\Pi_{ij}(x) = \{x_i, x_j\}. \quad (2.2)$$

If this matrix is invertible at each x , then Π is called nondegenerate or **symplectic**. In this case, the local matrices $(\omega_{ij}) = (-\Pi_{ij})^{-1}$ define a global 2-form $\omega \in \Omega^2(P) = \Gamma(\bigwedge^2 T^*P)$, and the condition $[\Pi, \Pi] = 0$ is equivalent to $d\omega = 0$.

Example 2.1 (Constant Poisson structures)

Let $P = \mathbb{R}^n$, and suppose that the $\Pi_{ij}(x)$ are constant. By a linear change of coordinates, one can find new coordinates $(q_1, \dots, q_k, p_1, \dots, p_k, e_1, \dots, e_l)$, $2k + l = n$, so that

$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

In terms of the bracket, we have

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

which is the original Poisson bracket in mechanics. In this example, all the coordinates e_i are Casimirs.

Example 2.2 (*Poisson structures on \mathbb{R}^2*)

Any smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defines a Poisson structure in $\mathbb{R}^2 = \{(x_1, x_2)\}$ by

$$\{x_1, x_2\} := f(x_1, x_2),$$

and every Poisson structure on \mathbb{R}^2 has this form.

Example 2.3 (*Lie-Poisson structures*)

An important class of Poisson structures are the linear ones. If P is a (finite-dimensional) vector space V considered as a manifold, with linear coordinates (x_1, \dots, x_n) , a linear Poisson structure is determined by constants c_{ij}^k satisfying

$$\{x_i, x_j\} = \sum_{k=1}^n c_{ij}^k x_k. \quad (2.3)$$

(We may assume that $c_{ij}^k = -c_{ji}^k$.) Such Poisson structures are usually called **Lie-Poisson structures**, since the Jacobi identity for the Poisson bracket implies that the c_{ij}^k are the structure constants of a Lie algebra \mathfrak{g} , which may be identified in a natural way with V^* . (Also, these Poisson structures were originally introduced by Lie [55] himself.) Note that we may also identify V with \mathfrak{g}^* . Conversely, any Lie algebra \mathfrak{g} with structure constants c_{ij}^k defines by (2.3) a linear Poisson structure on \mathfrak{g}^* .

Remark 2.4 (*Casimir functions*)

Deformation quantization of the Lie-Poisson structure on \mathfrak{g}^* , see e.g. [10, 44], leads to the universal enveloping algebra $U(\mathfrak{g})$. Elements of the center of $U(\mathfrak{g})$ are known as Casimir elements (or Casimir operators, when a representation of \mathfrak{g} is extended to a representation of $U(\mathfrak{g})$). These correspond to the center of the Poisson algebra of functions on \mathfrak{g}^* , hence, by extension, the designation “Casimir functions” for the center of any Poisson algebra.

2.2 Dirac structures

We now introduce a simultaneous generalization of Poisson structures and closed 2-forms. (We will often refer to closed 2-forms as **presymplectic**.)

Each 2-form ω on P corresponds to a bundle map

$$\tilde{\omega} : TP \rightarrow T^*P, \quad \tilde{\omega}(v)(u) = \omega(v, u). \quad (2.4)$$

Similarly, for a bivector field $\Pi \in \mathcal{X}^2(P)$, we define the bundle map

$$\tilde{\Pi} : T^*P \rightarrow TP, \quad \beta(\tilde{\Pi}(\alpha)) = \Pi(\alpha, \beta). \quad (2.5)$$

The matrix representing $\tilde{\Pi}$ in the bases (dx_i) and $(\partial/\partial x_i)$ corresponding to local coordinates induced by coordinates (x_1, \dots, x_n) on P is, up to a sign, just (2.2). So bivector fields (or 2-forms) are nondegenerate if and only if the associated bundle maps are invertible.

By using the maps in (2.4) and (2.5), we can describe both closed 2-forms and Poisson bivector fields as subbundles of $TP \oplus T^*P$: we simply consider the graphs

$$L_\omega := \text{graph}(\tilde{\omega}), \quad \text{and} \quad L_\Pi := \text{graph}(\tilde{\Pi}).$$

To see which subbundles of $TP \oplus T^*P$ are of this form, we introduce the following canonical structure on $TP \oplus T^*P$:

- 1) The symmetric bilinear form $\langle \cdot, \cdot \rangle_+ : TP \oplus T^*P \rightarrow \mathbb{R}$,

$$\langle (X, \alpha), (Y, \beta) \rangle_+ := \alpha(Y) + \beta(X). \quad (2.6)$$

- 2) The bracket $[[\cdot, \cdot]] : \Gamma(TP \oplus T^*P) \times \Gamma(TP \oplus T^*P) \rightarrow \Gamma(TP \oplus T^*P)$,

$$[[(X, \alpha), (Y, \beta)]] := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha). \quad (2.7)$$

Remark 2.5 (*Courant bracket*)

The bracket (2.7) is the non-skew-symmetric version, introduced in [56] (see also [79]), of T. Courant's original bracket [27]. The bundle $TP \oplus T^*P$ together with the brackets (2.6) and (2.7) is an example of a **Courant algebroid** [56].

Using the brackets (2.6) and (2.7), we have the following result [27]:

Proposition 2.6 *A subbundle $L \subset TP \oplus T^*P$ is of the form $L_\Pi = \text{graph}(\tilde{\Pi})$ (resp. $L_\omega = \text{graph}(\tilde{\omega})$) for a bivector field Π (resp. 2-form ω) if and only if*

*i) $TP \cap L = \{0\}$ (resp. $L \cap T^*P = \{0\}$) at all points of P ;*

ii) L is maximal isotropic with respect to $\langle \cdot, \cdot \rangle_+$;

furthermore, $[\Pi, \Pi] = 0$ (resp. $d\omega = 0$) if and only if

iii) $\Gamma(L)$ is closed under the Courant bracket (2.7).

Recall that L being isotropic with respect to $\langle \cdot, \cdot \rangle_+$ means that, at each point of P ,

$$\langle (X, \alpha), (Y, \beta) \rangle_+ = 0$$

whenever $(X, \alpha), (Y, \beta) \in L$. Maximality is equivalent to the dimension condition $\text{rank}(L) = \dim(P)$.

A **Dirac structure** on P is a subbundle $L \subset TP \oplus T^*P$ which is maximal isotropic with respect to $\langle \cdot, \cdot \rangle_+$ and whose sections are closed under the Courant bracket (2.7); in other words, a Dirac structure satisfies conditions *ii)* and *iii)* of Prop. 2.6 but is not necessarily the graph associated to a bivector field or 2-form.

If L satisfies only *ii)*, it is called an **almost Dirac structure**, and we refer to *iii)* as the **integrability condition** of a Dirac structure. The next example illustrates these notions in another situation.

Example 2.7 (*Regular foliations*)

Let $F \subseteq TP$ be a subbundle, and let $F^\circ \subset T^*P$ be its annihilator. Then $L = F \oplus F^\circ$ is an almost Dirac structure; it is a Dirac structure if and only if F satisfies the Frobenius condition

$$[\Gamma(F), \Gamma(F)] \subset \Gamma(F).$$

So regular foliations are examples of Dirac structures.

Example 2.8 (*Vector Dirac structures*)

If V is a finite-dimensional real vector space, then a **vector Dirac structure** on V is a subspace $L \subset V \oplus V^*$ which is maximal isotropic with respect to the symmetric pairing (2.6).¹

Let L be a vector Dirac structure on V . Let $\text{pr}_1 : V \oplus V^* \rightarrow V$ and $\text{pr}_2 : V \oplus V^* \rightarrow V^*$ be the canonical projections, and consider the subspace

$$R := \text{pr}_1(L) \subseteq V.$$

Then L induces a skew-symmetric bilinear form θ on R defined by

$$\theta(X, Y) := \alpha(Y), \tag{2.8}$$

where $X, Y \in R$ and $\alpha \in V^*$ is such that $(X, \alpha) \in L$.

Exercise

Show that θ is well defined, i.e., (2.8) is independent of the choice of α .

Conversely, any pair (R, θ) , where $R \subseteq V$ is a subspace and θ is a skew-symmetric bilinear form on R , defines a vector Dirac structure by

$$L := \{(X, \alpha), \ X \in R, \ \alpha \in V^* \text{ with } \alpha|_R = i_X \theta\}. \tag{2.9}$$

Exercise

Check that L defined in (2.9) is a vector Dirac structure on V with associated subspace R and bilinear form θ .

Example 2.8 indicates a simple way in which vector Dirac structures can be restricted to subspaces.

Example 2.9 (*Restriction of Dirac structures to subspaces*)

Let L be a vector Dirac structure on V , let $W \subseteq V$ be a subspace, and consider the pair (R, θ) associated with L . Then W inherits the vector Dirac structure L_W from L defined by the pair

$$R_W := R \cap W, \text{ and } \theta_W := \iota^* \theta,$$

where $\iota : W \hookrightarrow V$ is the inclusion map.

¹Vector Dirac structures are sometimes called “linear Dirac structures,” but we will eschew this name to avoid confusion with linear (i.e. Lie-) Poisson structures. (See Example 2.3)

Exercise

Show that there is a canonical isomorphism

$$L_W \cong \frac{L \cap (W \oplus V^*)}{L \cap W^\circ}. \quad (2.10)$$

Let (P, L) be a Dirac manifold, and let $\iota : N \hookrightarrow P$ be a submanifold. The construction in Example 2.9, when applied to $T_x N \subseteq T_x P$ for all $x \in P$, defines a maximal isotropic “subbundle” $L_N \subset TN \oplus T^*N$. The problem is that L_N may not be a continuous family of subspaces. When L_N is a continuous family, it is a smooth bundle which then automatically satisfies the integrability condition [27, Cor. 3.1.4], so L_N defines a Dirac structure on N .

The next example is a special case of this construction and is one of the original motivations for the study of Dirac structures; it illustrates the connection between Dirac structures and “constraint submanifolds” in classical mechanics.

Example 2.10 (*Momentum level sets*)

Let $J : P \rightarrow \mathfrak{g}^*$ be the momentum ² map for a hamiltonian action of a Lie group G on a Poisson manifold P [58]. Let $\mu \in \mathfrak{g}^*$ be a regular value for J , let G_μ be the isotropy group at μ with respect to the coadjoint action, and consider

$$Q = J^{-1}(\mu) \hookrightarrow P.$$

At each point $x \in Q$, we have a vector Dirac structure on $T_x Q$ given by

$$(L_Q)_x := \frac{L_x \cap (T_x Q \oplus T_x^* P)}{L_x \cap T_x Q^\circ}. \quad (2.11)$$

To show that L_Q defines a smooth bundle, it suffices to verify that $L_x \cap T_x Q^\circ$ has constant dimension. (Indeed, if this is the case, then $L_x \cap (T_x Q \oplus T_x^* P)$ has constant dimension as well, since the quotient $L_x \cap (T_x Q \oplus T_x^* P) / L_x \cap T_x Q^\circ$ has constant dimension, and this insures that all bundles are smooth.) A direct computation shows that $L_x \cap T_x Q^\circ$ has constant dimension if and only if the stabilizer groups of the G_μ -action on Q have constant dimension, which happens whenever the G_μ -orbits on Q have constant dimension (for instance, when the action of G_μ on Q is locally free). In this case, L_Q is a Dirac structure on Q .

We will revisit this example in Section 2.7.

Remark 2.11 (*Complex Dirac structures and generalized complex geometry*)

Using the natural extensions of the symmetric form (2.6) and the Courant bracket (2.7) to $(TP \oplus T^*P) \otimes \mathbb{C}$, one can define a **complex Dirac structure** on a manifold P to be a maximal isotropic *complex* subbundle $L \subset (TP \oplus T^*P) \otimes \mathbb{C}$ whose sections are closed under the Courant bracket. If a complex Dirac structure L satisfies the condition

$$L \cap \overline{L} = \{0\} \quad (2.12)$$

at all points of P (here \overline{L} is the complex conjugate of L), then it is called a **generalized complex structure**; such structures were introduced in [42, 45] as a common generalization of complex and symplectic structures.

²The term “moment” is frequently used instead of “momentum” in this context. In this paper, we will follow the convention, introduced in [60], that “moment” is used only in connection with groupoid actions. As we will see (e.g. in Example 4.16), many momentum maps, even for “exotic” theories, are moment maps as well.

To see how complex structures fit into this picture, note that an almost complex structure $J : TP \rightarrow TP$ defines a maximal isotropic subbundle $L_J \subset (TP \oplus T^*P) \otimes \mathbb{C}$ as the i -eigenbundle of the map

$$(TP \oplus T^*P) \otimes \mathbb{C} \rightarrow (TP \oplus T^*P) \otimes \mathbb{C}, \quad (X, \alpha) \mapsto (-J(X), J^*(\alpha)).$$

The bundle L_J completely characterizes J , and it satisfies (2.12); moreover, L_J satisfies the integrability condition of a Dirac structure if and only if J is a complex structure.

Similarly, a symplectic structure ω on P can be seen as a generalized complex structure through the bundle $L_{\omega, \mathbb{C}}$, defined as the i -eigenbundle of the map

$$(TP \oplus T^*P) \otimes \mathbb{C} \rightarrow (TP \oplus T^*P) \otimes \mathbb{C}, \quad (X, \alpha) \mapsto (\tilde{\omega}(X), -\tilde{\omega}^{-1}(\alpha)).$$

Note that, by (2.12), a generalized complex structure is never the complexification of a real Dirac structure. In particular, for a symplectic structure ω , $L_{\omega, \mathbb{C}}$ is *not* the complexification of the real Dirac structure L_ω of Proposition 2.6.

2.3 Twisted structures

A “background” *closed* 3-form $\phi \in \Omega^3(P)$ can be used to “twist” the geometry of P [47, 68], leading to a modified notion of Dirac structures [79], and in particular of Poisson structure. The key point is to use ϕ to alter the ordinary Courant bracket (2.7) as follows:

$$[(X, \alpha), (Y, \beta)]_\phi := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + \phi(X, Y, \cdot)). \quad (2.13)$$

We now simply repeat the definitions in Section 2.2 replacing (2.7) by the **ϕ -twisted Courant bracket** (2.13).

A **ϕ -twisted Dirac structure** on P is a subbundle $L \subset TP \oplus T^*P$ which is maximal isotropic with respect to $\langle \cdot, \cdot \rangle_+$ (2.6) and for which

$$[[\Gamma(L), \Gamma(L)]_\phi] \subseteq \Gamma(L). \quad (2.14)$$

With this new integrability condition, one can check that the graph of a bivector field Π is a ϕ -twisted Dirac structure if and only if

$$\frac{1}{2}[\Pi, \Pi] = \wedge^3 \tilde{\Pi}(\phi);$$

such bivector fields are called **ϕ -twisted Poisson structures**. Similarly, the graph of a 2-form ω is a ϕ -twisted Dirac structure if and only if

$$d\omega + \phi = 0,$$

in which case ω is called a **ϕ -twisted presymplectic structure**.

Remark 2.12 (*Terminology*)

The term “twisted Dirac structure” and its cousins represent a certain abuse of terminology, since it is not the Dirac (or Poisson, etc.) structure which is twisted, but rather the *notion* of Dirac structure. Nevertheless, we have chosen to stick to this terminology, rather than the alternative “Dirac structure with background” [49], because it is consistent with such existing terms as “twisted sheaf”, and because the alternative terms lead to some awkward constructions.

Example 2.13 (*Cartan-Dirac structures on Lie groups*)

Let G be a Lie group whose Lie algebra \mathfrak{g} is equipped with a nondegenerate adjoint-invariant symmetric bilinear form $(\cdot, \cdot)_{\mathfrak{g}}$, which we use to identify TG and T^*G . In $TG \oplus TG \sim TG \oplus T^*G$, we consider the maximal isotropic subbundle

$$L_G := \{(v_r - v_l, \frac{1}{2}(v_r + v_l)), \ v \in \mathfrak{g}\}, \quad (2.15)$$

where v_r and v_l are the right and left invariant vector fields corresponding to v . One can show that L_G is a ϕ^G -twisted Dirac structure, where ϕ^G is the bi-invariant Cartan 3-form on G , defined on Lie algebra elements by

$$\phi^G(u, v, w) = \frac{1}{2}(u, [v, w])_{\mathfrak{g}}.$$

We call L_G the **Cartan-Dirac structure** on G associated with $(\cdot, \cdot)_{\mathfrak{g}}$. Note that L_G is of the form L_{Π} only at points g for which $\text{Ad}_g + 1$ is invertible, see also Example 2.19.

These Dirac structures are closely related to the theory of quasi-hamiltonian spaces and group-valued momentum maps [3, 15, 96], as well as to quasi-Poisson manifolds [2, 14].

2.4 Symplectic leaves and local structure of Poisson manifolds

If Π is a symplectic Poisson structure on P , then Darboux's theorem asserts that, around each point of P , one can find coordinates $(q_1, \dots, q_k, p_1, \dots, p_k)$ such that

$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

The corresponding symplectic form ω is

$$\omega = \sum_i dq_i \wedge dp_i.$$

In general, the image of the bundle map (2.5), $\tilde{\Pi}(T^*P) \subseteq TP$, defines an integrable singular distribution on P ; in other words, P is a disjoint union of “leaves” \mathcal{O} satisfying $T_x \mathcal{O} = \tilde{\Pi}(T_x^*P)$ for all $x \in P$. The leaf \mathcal{O} through x can be described as the points which can be reached from x through piecewise hamiltonian paths.

If $\tilde{\Pi}$ has locally constant rank, we call the Poisson structure Π **regular**, in which case it defines a foliation of P in the ordinary sense. Note that this is always the case on an open dense subset of P , called the **regular part**.

The local structure of a Poisson manifold (P, Π) around a regular point is given the Lie-Darboux theorem: If Π has constant rank k around a given point, then there exist coordinates $(q_1, \dots, q_k, p_1, \dots, p_k, e_1, \dots, e_l)$ such that

$$\{q_i, p_j\} = \delta_{ij}, \text{ and } \{q_i, q_j\} = \{p_i, p_j\} = \{q_i, e_j\} = \{p_i, e_j\} = 0.$$

Thus, the local structure of a regular Poisson manifold is determined by that of the vector Poisson structures on any of its tangent spaces (in a given connected component).

In the general case, we have the local splitting theorem [87]:

Theorem 2.14 *Around any point x_0 in a Poisson manifold P , there exist coordinates*

$$(q_1, \dots, q_k, p_1, \dots, p_k, e_1, \dots, e_l), \quad (q, p, e)(x_0) = (0, 0, 0),$$

such that

$$\Pi = \sum_{i=1}^k \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^l \eta_{ij}(e) \frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial e_j}$$

and $\eta_{ij}(0) = 0$.

The splitting of Theorem 2.14 has a symplectic factor associated with the coordinates (q_i, p_i) and a totally degenerate factor (i.e., with all Poisson brackets vanishing at $e = 0$) associated with the coordinates (e_j) . The symplectic factor may be identified with an open subset of the leaf \mathcal{O} through x_0 ; patching them together defines a symplectic structure on each leaf of the foliation determined by Π . So Π canonically defines a singular **symplectic foliation** of P . The totally degenerate factor in the local splitting is well-defined up to isomorphism. Its isomorphism class is the same at all points in a given symplectic leaf, so one refers to the totally degenerate factor as the **transverse structure** to Π along the leaf.

Example 2.15 (*Symplectic leaves of Poisson structures on \mathbb{R}^2*)

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function, and let us consider the Poisson structure on $\mathbb{R}^2 = \{(x_1, x_2)\}$ defined by

$$\{x_1, x_2\} := f(x_1, x_2).$$

The connected components of the set where $f(x_1, x_2) \neq 0$ are the 2-dimensional symplectic leaves; in the set where f vanishes, each point is a symplectic leaf.

Example 2.16 (*Symplectic leaves of Lie-Poisson structures*)

Let us consider \mathfrak{g}^* , the dual of the Lie algebra \mathfrak{g} , equipped with its Lie-Poisson structure, see Example 2.3. The symplectic leaves are just the coadjoint orbits for any connected group with Lie algebra \mathfrak{g} . Since $\{0\}$ is always an orbit, a Lie-Poisson structure is not regular unless \mathfrak{g} is abelian.

Exercise

Describe the symplectic leaves in the duals of $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{a}(1)$ (nonabelian 2-dimensional Lie algebra).

Remark 2.17 (*Linearization problem*)

By linearizing at x_0 the functions η_{ij} in Theorem 2.14, we can write

$$\{e_i, e_j\} = \sum_k c_{ij}^k e_k + O(e^2), \tag{2.16}$$

and it turns out that c_{ij}^k define a Lie-Poisson structure on the normal space to the symplectic leaf at x_0 . The **linearization problem** consists of determining whether one can choose suitable “transverse” coordinates (e_1, \dots, e_l) with respect to which $O(e^2)$ in (2.16) vanishes. For example, if the Lie algebra structure on the conormal bundle to a symplectic leaf determined by the linearization of Π at a point x_0 is semi-simple and of compact type, then Π is linearizable

around x_0 through a smooth change of coordinates. The first proof of this theorem, due to Conn [25], used many estimates and a “hard” implicit function theorem. A “soft” proof, using only the sort of averaging usually associated with compact group actions (but for groupoids instead of groups), has recently been announced by Crainic and Fernandes [31]. There is also a “semilocal” problem of linearization in the neighborhood of an entire symplectic leaf. This problem was first addressed by Vorobjev [85], with further developments by Davis and Wade [32]. For overviews of linearization and more general normal form questions, we refer to the article of Fernandes and Monnier [37] and the forthcoming book of Dufour and Zung [34].

2.5 Presymplectic leaves and Poisson quotients of Dirac manifolds

Let $\text{pr}_1 : TP \oplus T^*P \rightarrow TP$ and $\text{pr}_2 : TP \oplus T^*P \rightarrow T^*P$ be the canonical projections. If $L \subset TP \oplus T^*P$ is a (twisted) Dirac structure on P , then

$$\text{pr}_1(L) \subseteq TP \quad (2.17)$$

defines a singular distribution on P . Note that if $L = L_\Pi$ for a Poisson structure Π , then $\text{pr}_1(L) = \tilde{\Pi}(T^*P)$, so this distribution coincides with the one defined by Π , see Section 2.4. It turns out that the integrability condition for (twisted) Dirac structures guarantees that (2.17) is integrable in general, so a (twisted) Dirac structure L on P determines a decomposition of P into leaves \mathcal{O} satisfying

$$T_x \mathcal{O} = \text{pr}_1(L)_x$$

at all $x \in P$.

Just as leaves of foliations associated with Poisson structures carry symplectic forms, each leaf of a (twisted) Dirac manifold P is naturally equipped with a (twisted) presymplectic 2-form θ : at each $x \in P$, θ_x is the bilinear form defined in (2.8). These forms fit together into a smooth leafwise 2-form, which is nondegenerate on the leaves just when L is a (twisted) Poisson structure. If L is twisted by ϕ , then θ is twisted by the pull back of ϕ to each leaf.

Remark 2.18 (*Lie algebroids*)

The fact that $\text{pr}_1(L) \subseteq TP$ is an integrable singular distribution is a consequence of a more general fact: the restriction of the Courant bracket $\llbracket \cdot, \cdot \rrbracket_\phi$ to $\Gamma(L)$ defines a Lie algebra bracket making $L \rightarrow P$ into a *Lie algebroid* with anchor $\text{pr}_1|_L$, and the image of the anchor of any Lie algebroid is always an integrable distribution (its leaves are also called **orbits**). We refer to [20, 62] for more on Lie algebroids.

Example 2.19 (*Presymplectic leaves of Cartan-Dirac structures*)

Let L_G be a Cartan-Dirac structure on G with respect to $(\cdot, \cdot)_{\mathfrak{g}}$, see (2.15). Then the associated distribution on G is

$$\text{pr}_1(L_G) = \{v_r - v_l, v \in \mathfrak{g}\}.$$

Since vector fields of the form $v_r - v_l$ are infinitesimal generators of the action of G on itself by conjugation, it follows that the twisted presymplectic leaves of L_G are the connected components of the conjugacy classes of G . With $v_G = v_r - v_l$, the corresponding twisted presymplectic forms can be written as

$$\theta_g(v_G, w_G) := \frac{1}{2}((\text{Ad}_{g^{-1}} - \text{Ad}_g)v, w)_{\mathfrak{g}}, \quad (2.18)$$

at $g \in G$. These 2-forms were introduced in [43] in the study of the symplectic structure of certain moduli spaces. They are analogous to the Kostant-Kirillov-Souriau symplectic forms on

coadjoint orbits, although they are neither nondegenerate nor closed: θ_g is degenerate whenever $1 + \text{Ad}_g$ is not invertible, and, on a conjugacy class $\iota : \mathcal{O} \hookrightarrow G$, $d\theta = -\iota^*\phi^G$.

Just as the symplectic forms along coadjoint orbits on the dual of a Lie algebra are associated with Lie-Poisson structures, the 2-forms (2.18) along conjugacy classes of a Lie group are associated with Cartan-Dirac structures.

For any ϕ -twisted Dirac structure L , the (topologically) closed family of subspaces $TP \cap L = \ker(\theta)$ in TP is called the **characteristic distribution** of L and is denoted by $\ker(L)$. It is always contained in $\text{pr}_1(L)$. When $\ker(L)$ has constant fibre dimension, it is integrable if and only if

$$\phi(X, Y, Z) = 0 \quad \text{for all } X, Y \in \ker(\theta), Z \in \text{pr}_1(L), \quad (2.19)$$

at each point of P . In this case, the leaves of the corresponding **characteristic foliation** are the null spaces of the presymplectic forms along the leaves. On each leaf $\iota : \mathcal{O} \hookrightarrow P$, the 2-form θ is basic with respect to the characteristic foliation if and only if

$$\ker(\theta) \subseteq \ker(\iota^*\phi) \quad (2.20)$$

at all points of \mathcal{O} . In this case, forming the leaf space of this foliation (locally, or globally when the foliation is simple) produces a quotient manifold bearing a singular foliation by twisted symplectic leaves; it is in fact a twisted Poisson manifold. In particular, when $\phi = 0$, conditions (2.19) and (2.20) are satisfied, and the quotient is an ordinary Poisson manifold. Thus, Dirac manifolds can be regarded as “pre-Poisson” manifolds, since, in nice situations, they become Poisson manifolds after they are divided out by the characteristic foliation.

Functions which are annihilated by all tangent vectors in the characteristic distribution (equivalently, have differentials in the projection of L to T^*P) are called **admissible** [27]. For admissible f and g , one can define

$$\{f, g\} := \theta(X_f, X_g), \quad (2.21)$$

where X_f is any vector field such that $(X_f, df) \in L$. (Note that any two choices for X_f differ by a characteristic vector, so the bracket (2.21) is well defined.) If (2.20) holds, then the algebra of admissible functions is closed under this bracket, but it is not in general a Poisson algebra, due to the presence of ϕ . In particular, if the characteristic foliation is regular and simple, the admissible functions are just the functions on the (twisted) Poisson quotient.

Example 2.20 (*Nonintegrable characteristic distributions*)

Consider the presymplectic structure $x_1 dx_1 \wedge dx_2$ on \mathbb{R}^2 . Its characteristic distribution consists of the zero subspace at points where $x_1 \neq 0$ and the entire tangent space at each point of the x_2 axis. Thus, the points off the axis are integral manifolds, while there are no integral manifolds through points on the axis. The only admissible functions are constants.

On the other hand, if a 2-form is not closed, then its kernel may have constant fibre dimension and still be nonintegrable. For example, the characteristic distribution of the 2-form $x_2 dx_1 \wedge dx_4 - dx_3 \wedge dx_4$ on \mathbb{R}^4 is spanned by $\partial/\partial x_1 + x_2 \partial/\partial x_3$ and $\partial/\partial x_2$. A direct computation shows that this 2-dimensional distribution does not satisfy the Frobenius condition, so it is not integrable.

Example 2.21 (*A nonreducible 2-form*)

The characteristic foliation of the 2-form $(x_3^2 + 1)dx_1 \wedge dx_2$ on \mathbb{R}^3 consists of lines parallel to the x_3 -axis, so it is simple. However, the form is not basic with respect to this foliation.

We will say more about presymplectic leaves and quotient Poisson structures in Section 2.7.

2.6 Poisson maps

Although we shall see later that the following notion of morphism between Poisson manifolds is not the only one, it is certainly the most obvious one.

Let (P_1, Π_1) and (P_2, Π_2) be Poisson manifolds. A smooth map $\psi : P_1 \rightarrow P_2$ is a **Poisson map** if $\psi^* : C^\infty(P_2) \rightarrow C^\infty(P_1)$ is a homomorphism of Poisson algebras, i.e.,

$$\{f, g\}_2 \circ \psi = \{f \circ \psi, g \circ \psi\}_1$$

for $f, g \in C^\infty(P_2)$. One can reformulate this condition in terms of Poisson bivectors or hamiltonian vector fields as follows. A map $\psi : P_1 \rightarrow P_2$ is a Poisson map if and only if either of the following two equivalent conditions hold:

- i) $\psi_* \Pi_1 = \Pi_2$, i.e., Π_1 and Π_2 are ψ -related.
- ii) For all $f \in C^\infty(P_2)$, $X_f = \psi_*(X_{\psi^* f})$.

It is clear by condition *ii*) that trajectories of $X_{\psi^* f}$ project to those of X_f if ψ is a Poisson map. This provides a way to “lift” some paths from P_2 to P_1 . However, knowing that X_f is complete does not guarantee that $X_{\psi^* f}$ is complete. In order to assure that there are no “missing points” on the lifted trajectory on P_1 , we define a Poisson map $\psi : P_1 \rightarrow P_2$ to be **complete** if for any $f \in C^\infty(P_2)$ such that X_f is complete, then $X_{\psi^* f}$ is also complete. Alternatively, one can replace the condition of X_f being complete by X_f (or f itself) having compact support. Note that there is no notion of completeness (or “missing point”) for a Poisson manifold by itself, only for a Poisson manifold *relative* to another.

Remark 2.22 (*Cotangent paths*)

The path lifting alluded to above is best understood in terms of so-called cotangent paths [39, 90]. A **cotangent path** on a Poisson manifold P is a path α in T^*P such that $(\pi \circ \alpha)' = \tilde{\Pi} \circ \alpha$, where π is the cotangent bundle projection. If $\psi : P_1 \rightarrow P_2$ is a Poisson map, then a cotangent path α_1 on P_1 is a **horizontal lift** of the cotangent path α_2 on P_2 if $\alpha_1(t) = \psi^*(\alpha_2(t))$ for all t . It turns out that a cotangent path on P_2 has at most one horizontal lift for each initial value of $\pi \circ \alpha_1$. Furthermore, the existence of horizontal lifts for all cotangent paths α_2 and all initial data consistent with the initial value of α_2 is equivalent to completeness of the map ψ .

This path lifting property suggests that complete Poisson maps play the role of “coverings” in Poisson geometry. This idea is borne out by some of the examples below.

Example 2.23 (*Complete functions*)

Let us regard \mathbb{R} as a Poisson manifold, equipped with the zero Poisson bracket. (This is the only possible Poisson structure on \mathbb{R} .) Then any map $f : P \rightarrow \mathbb{R}$ is a Poisson map, which is complete if and only if X_f is a complete vector field.

Observe that the notion of completeness singles out the subset of $C^\infty(P)$ consisting of complete functions, which is preserved under complete Poisson maps.

Exercise

For which Poisson manifolds is the set of complete functions closed under addition? (Hint: when are all functions complete?)

Example 2.24 (*Open subsets of symplectic manifolds*)

Let (P, Π) be a symplectic manifold, and let $U \subseteq P$ be an open subset. Then the inclusion map $U \hookrightarrow P$ is complete if and only if U is closed (hence a union of connected components). More generally, the image of a complete Poisson map is a union of symplectic leaves.

Example 2.24 suggests that (connected) symplectic manifolds are “minimal objects” among Poisson manifolds.

Exercise

The inclusion of every symplectic leaf in a Poisson manifold is a complete Poisson map.

Exercise

Let P_1 be a Poisson manifold, and let P_2 be symplectic. Then any Poisson map $\psi : P_1 \rightarrow P_2$ is a submersion. Furthermore, if P_2 is connected and ψ is complete, then ψ is surjective (assuming that P_1 is nonempty).

The previous exercise is the first step in establishing that complete Poisson maps with symplectic target must be fibrations. In fact, if P_1 is symplectic and $\dim(P_1) = \dim(P_2)$, then a complete Poisson map $\psi : P_1 \rightarrow P_2$ is a covering map. In general, a complete Poisson map $\psi : P_1 \rightarrow P_2$, where P_2 is *symplectic*, is a locally trivial symplectic fibration with a flat Ehresmann connection: the horizontal lift in $T_x P_1$ of a vector X in $T_{\psi(x)} P_2$ is defined as

$$\tilde{\Pi}_1((T_x \psi)^* \tilde{\Pi}_2^{-1}(X)).$$

The horizontal subspaces define a foliation whose leaves are coverings of P_2 , and P_1 and ψ are completely determined, up to isomorphism, by the holonomy

$$\pi_1(P_2, x) \rightarrow \text{Aut}(\psi^{-1}(x)),$$

see [20, Sec. 7.6] for details.

2.7 Dirac maps

To see how to define Dirac maps, we first reformulate the condition for a map $\psi : (P_1, \Pi_1) \rightarrow (P_2, \Pi_2)$ to be Poisson in terms of the bundles $L_{\Pi_1} = \text{graph}(\tilde{\Pi}_1)$ and $L_{\Pi_2} = \text{graph}(\tilde{\Pi}_2)$. First, note that ψ is a Poisson map if and only if, at each $x \in P_1$,

$$(\tilde{\Pi}_2)_{\psi(x)} = T_x \psi \circ (\tilde{\Pi}_1)_x \circ (T_x \psi)^*. \quad (2.22)$$

Now, using (2.22), it is not difficult to check that ψ being a Poisson map is equivalent to

$$L_{\Pi_2} = \{(T\psi(X), \beta) \mid X \in TP_1, \beta \in T^*P_2, (X, (T\psi)^*(\beta)) \in L_{\Pi_1}\}. \quad (2.23)$$

Similarly, if (P_1, ω_1) and (P_2, ω_2) are presymplectic manifolds, then a map $\psi : P_1 \rightarrow P_2$ satisfies $\psi^* \omega_2 = \omega_1$ if and only if L_{ω_1} and L_{ω_2} are related by

$$L_{\omega_1} = \{(X, (T\psi)^*(\beta)) \mid X \in TP_1, \beta \in TP_2, (T\psi(X), \beta) \in L_{\omega_2}\}. \quad (2.24)$$

Since Dirac structures simultaneously generalize Poisson structures and presymplectic forms, and conditions (2.23) and (2.24) both make sense for arbitrary Dirac subbundles, we have *two*

possible definitions: If (P_1, L_1) and (P_2, L_2) are (possibly twisted) Dirac manifolds, then a map $\psi : P_1 \rightarrow P_2$ is a **forward Dirac map** if

$$L_2 = \{(T\psi(X), \beta) \mid X \in TP_1, \beta \in T^*P_2, (X, (T\psi)^*(\beta)) \in L_1\}, \quad (2.25)$$

and a **backward Dirac map** if

$$L_1 = \{(X, (T\psi)^*(\beta)) \mid X \in TP_1, \beta \in TP_2, (T\psi(X), \beta) \in L_2\}. \quad (2.26)$$

Regarding vector Dirac structures as *odd* (in the sense of super geometry) analogues of lagrangian subspaces, one can interpret formulas (2.25) and (2.26) via composition of canonical relations [86], see [16].

The expression on the right-hand side of (2.25) defines at each point of P_1 a way to push a Dirac structure forward, whereas (2.26) defines a pull-back operation. For this reason, we often write

$$L_2 = \psi_* L_1$$

when (2.25) holds, following the notation for ψ -related vector or bivector fields; similarly, we may write

$$L_1 = \psi^* L_2$$

instead of (2.26). This should explain the terminology “forward” and “backward”.

Remark 2.25 (*Isotropic and coisotropic subspaces*)

The notions of isotropic and coisotropic subspaces, as well as much of the usual lagrangian/coisotropic calculus [86, 89] can be naturally extended to Dirac vector spaces. This yields an alternative characterization of forward (resp. backward) Dirac maps in terms of their graphs being coisotropic (resp. isotropic) subspaces of the suitable product Dirac space [83].

Note that the pointwise pull back $\psi^* L_2$ is always a well-defined family of maximal isotropic subspaces in the fibres of $TP_1 \oplus T^*P_1$, though it may not be continuous, whereas $\psi_* L_1$ may not be well-defined at all.

Exercise

Consider a smooth map $f : P_1 \rightarrow P_2$, and let L_2 be a ϕ -twisted Dirac structure on P_2 . Show that if $f^* L_2$ defines a smooth vector bundle, then its sections are automatically closed under the $f^* \phi$ -twisted Courant bracket on P_1 (so that $f^* L_2$ is a $f^* \phi$ -twisted Dirac structure).

If P_1 and P_2 are symplectic manifolds, then a map $\psi : P_1 \rightarrow P_2$ is forward Dirac if and only if it is a Poisson map, and backward Dirac if and only if it pulls back the symplectic form on P_2 to the one on P_1 , in which case we call it a **symplectic map**.

The next example shows that forward Dirac maps need not be backward Dirac, and vice versa.

Example 2.26 (*Forward vs. backward Dirac maps*)

Consider $\mathbb{R}^2 = \{(q, p)\}$, equipped with the symplectic form $dq \wedge dp$, and $\mathbb{R}^4 = \{(q_1, p_1, q_2, p_2)\}$, with symplectic form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. Then a simple computation shows that the inclusion

$$\mathbb{R}^2 \hookrightarrow \mathbb{R}^4, \quad (q, p) \mapsto (q, p, 0, 0),$$

is a symplectic (i.e. backward Dirac) map, but it does not preserve Poisson brackets. On the other hand, the projection

$$\mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad (q_1, p_1, q_2, p_2) \mapsto (q_1, p_1),$$

is a Poisson (i.e. forward Dirac) map, but it is not symplectic.

Example 2.27 (*Backward Dirac maps and restrictions*)

Let (P, L) be a (possibly twisted) Dirac manifold, and let $\iota : N \hookrightarrow P$ be a submanifold. Let $L_N \subset TN \oplus T^*N$ be the subbundle defined pointwise by the restriction of L to N , see (2.10), and suppose that L_N is smooth, so that it defines a Dirac structure on N . A direct computation shows that

$$L_N = \iota^* L,$$

hence the inclusion ι is a backward Dirac map.

The next exercise explains when the notions of forward and backward Dirac maps coincide.

Exercise

Let V_1 and V_2 be vector spaces, and let $f : V_1 \rightarrow V_2$ be a linear map.

1. Let L be a vector Dirac structure on V_1 . Then $f^* f_* L = L$ if and only if $\ker(f) \subseteq \ker(L)$, where $\ker(L) = V \cap L$.
2. Let L be a vector Dirac structure on V_2 . Then $f_* f^* L = L$ if and only if $f(V_1) \subseteq R$, where $R = \text{pr}_1(L) \subseteq V_2$.

It follows that $f^* f_*(L) = L$ for all L if and only if f is injective, and $f_* f^*(L) = L$ for all L if and only if f is surjective.

In particular, the previous exercise shows that if P_1 and P_2 are symplectic manifolds, then a Poisson map $P_1 \rightarrow P_2$ is symplectic if and only if it is an immersion, and a symplectic map $P_1 \rightarrow P_2$ is Poisson if and only if the map is a submersion (compare with Example 2.26). Thus, the only maps which are both symplectic and Poisson are local diffeomorphisms.

Using the previous exercise, we find important examples of maps which are both forward and backward Dirac.

Example 2.28 (*Inclusion of presymplectic leaves*)

Let (P, L) be a twisted Dirac manifold. Let (\mathcal{O}, θ) be a presymplectic leaf, and let $\iota : \mathcal{O} \hookrightarrow P$ be the inclusion. We regard \mathcal{O} as a Dirac manifold, with Dirac structure $L_\theta = \text{graph}(\tilde{\theta})$. Then it follows from the definition of θ that ι is a backward Dirac map. On the other hand, since

$$T\iota(T\mathcal{O}) = \text{pr}_1(L)$$

at each point, $\iota_* L_\theta = \iota_* \iota^* L = L$, so ι is also a forward Dirac map.

Note that θ is completely determined by either of the conditions that the inclusion be forward or backward Dirac.

Example 2.29 (*Quotient Poisson structures*)

Let (P, L) be a Dirac manifold, and suppose that its characteristic foliation is regular and simple. According to the discussion in Section 2.5, the leaf space P_{red} has an induced Poisson structure Π_{red} . Using the definition of Π_{red} , one can directly show that the natural projection

$$\text{pr} : P \longrightarrow P_{\text{red}}$$

is a forward Dirac map, i.e., $\text{pr}_* L = \text{graph}(\tilde{\Pi}_{\text{red}})$. But since

$$\ker(T\text{pr}) = \ker(L),$$

the previous exercise implies that $\text{pr}^* \text{pr}_* L = L$, so pr is a backward Dirac map as well.

As in Example 2.28, Π_{red} is uniquely determined by either of the conditions that pr be backward or forward Dirac.

Example 2.29 has an important particular case, which illustrates the connection between Dirac geometry and the theory of hamiltonian actions.

Example 2.30 (*Poisson reduction*)

Suppose that $J : P \rightarrow \mathfrak{g}^*$ is the momentum map for a hamiltonian action of a Lie group G on a Poisson manifold (P, Π) . Let $\mu \in \mathfrak{g}^*$ be a regular value for J , let $Q = J^{-1}(\mu)$, and assume that the orbit space

$$P_{red} = Q/G_\mu$$

is a smooth manifold such that the projection $Q \rightarrow P_{red}$ is a surjective submersion. Following Examples 2.10 and 2.27, we know that Q has an induced Dirac structure L_Q with respect to which the inclusion $Q \hookrightarrow P$ is a backward Dirac map.

Exercise

Show that the G_μ -orbits on Q coincide with the characteristic foliation of L_Q .

Thus, by Example 2.29, P_{red} inherits a Poisson structure Π_{red} for which the projection $Q \rightarrow P_{red}$ is both backward and forward Dirac (and either one of these conditions defines Π_{red} uniquely).

3 Algebraic Morita equivalence

There is another notion of morphism between Poisson manifolds which, though it does *not* include all the Poisson maps, is more closely adapted to the “representation theory” of Poisson manifolds. It is based on an algebraic idea which we present first. (The impatient reader may skip to Section 4.)

3.1 Ring-theoretic Morita equivalence

Let \mathcal{A} and \mathcal{B} be unital algebras over a fixed ground ring k , and let ${}_{\mathcal{A}}\mathfrak{M}$ and ${}_{\mathcal{B}}\mathfrak{M}$ denote the categories of left modules over \mathcal{A} and \mathcal{B} , respectively. We call \mathcal{A} and \mathcal{B} **Morita equivalent** [64] if they have equivalent categories of left modules, i.e., if there exist functors

$$\mathcal{F} : {}_{\mathcal{B}}\mathfrak{M} \longrightarrow {}_{\mathcal{A}}\mathfrak{M} \quad \text{and} \quad \tilde{\mathcal{F}} : {}_{\mathcal{A}}\mathfrak{M} \longrightarrow {}_{\mathcal{B}}\mathfrak{M} \quad (3.1)$$

whose compositions are naturally equivalent to the identity functors:

$$\mathcal{F} \circ \tilde{\mathcal{F}} \cong \text{Id}_{{}_{\mathcal{A}}\mathfrak{M}}, \quad \text{and} \quad \tilde{\mathcal{F}} \circ \mathcal{F} \cong \text{Id}_{{}_{\mathcal{B}}\mathfrak{M}}.$$

One way to construct such functors between module categories is via bimodules: if ${}_A X_B$ is an $(\mathcal{A}, \mathcal{B})$ -bimodule (i.e., X is a k -module which is a left \mathcal{A} -module and a right \mathcal{B} -module, and these actions commute), then we define an associated functor $\mathcal{F}_X : {}_{\mathcal{B}}\mathfrak{M} \rightarrow {}_{\mathcal{A}}\mathfrak{M}$ by setting, at the level of objects,

$$\mathcal{F}_X({}_{\mathcal{B}}M) := {}_A X_B \otimes_B M. \quad (3.2)$$

where the \mathcal{A} -module structure on $\mathcal{F}_X({}_{\mathcal{B}}M)$ is given by

$$a \cdot (x \otimes_B m) = (ax) \otimes_B m.$$

For a morphism $T : {}_{\mathcal{B}}M \rightarrow {}_{\mathcal{B}}M'$, we define

$$\mathcal{F}_X(T) : {}_A X_B \otimes_B M \rightarrow {}_A X_B \otimes_B M', \quad \mathcal{F}_X(T)(x \otimes_B m) = x \otimes_B T(m). \quad (3.3)$$

This way of producing functors turns out to be very general. In fact, as we will see in Theorem 3.1, any functor establishing an equivalence between categories of modules is naturally equivalent to a functor associated with a bimodule.

Exercise

Let X and X' be $(\mathcal{A}, \mathcal{B})$ -bimodules. Show that the associated functors \mathcal{F}_X and $\mathcal{F}_{X'}$ are naturally equivalent if and only if the bimodules X and X' are isomorphic.

It follows from the previous exercise that the functors $\mathcal{F}_X : {}_{\mathcal{B}}\mathfrak{M} \rightarrow {}_{\mathcal{A}}\mathfrak{M}$, associated with an $(\mathcal{A}, \mathcal{B})$ -bimodule X , and $\mathcal{F}_Y : {}_{\mathcal{A}}\mathfrak{M} \rightarrow {}_{\mathcal{B}}\mathfrak{M}$, associated with an $(\mathcal{B}, \mathcal{A})$ -bimodule Y , are inverses of one another if and only if

$${}_{\mathcal{A}}X_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}Y_{\mathcal{A}} \cong \mathcal{A} \quad \text{and} \quad {}_{\mathcal{B}}Y_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}X_{\mathcal{B}} \cong \mathcal{B}. \quad (3.4)$$

The isomorphisms in (3.4) are *bimodule* isomorphisms, and \mathcal{A} and \mathcal{B} are regarded as $(\mathcal{A}, \mathcal{A})$ - and $(\mathcal{B}, \mathcal{B})$ -bimodules, respectively, in the natural way (with respect to left and right multiplications). So Morita equivalence is equivalent to the existence of bimodules satisfying (3.4).

One can see Morita equivalence as the notion of isomorphism in an appropriate category. For that, we think of an *arbitrary* $(\mathcal{A}, \mathcal{B})$ -bimodule as a “generalized morphism” between \mathcal{B} and \mathcal{A} . Note that, if $\mathcal{A} \xrightarrow{q} \mathcal{B}$ is an ordinary algebra homomorphism, then we can use it to make \mathcal{A} into an $(\mathcal{A}, \mathcal{B})$ -bimodule by

$$a \cdot x \cdot b := axq(b), \quad a \in \mathcal{A}, \quad x \in \mathcal{A}, \quad b \in \mathcal{B}. \quad (3.5)$$

Since the tensor product

$${}_{\mathcal{A}}X_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}Y_{\mathcal{C}}$$

is an $(\mathcal{A}, \mathcal{C})$ -bimodule, we can see it as a “composition” of bimodules. As this composition is only associative up to isomorphism, we consider the collection of *isomorphism classes* of $(\mathcal{A}, \mathcal{B})$ -bimodules, denoted by $\text{Bim}(\mathcal{A}, \mathcal{B})$. Then $\otimes_{\mathcal{B}}$ defines an associative composition

$$\text{Bim}(\mathcal{A}, \mathcal{B}) \times \text{Bim}(\mathcal{B}, \mathcal{C}) \rightarrow \text{Bim}(\mathcal{A}, \mathcal{C}). \quad (3.6)$$

We define the category Alg to be that in which the objects are unital k -algebras and the morphisms $\mathcal{A} \leftarrow \mathcal{B}$ are the isomorphism classes of $(\mathcal{A}, \mathcal{B})$ -bimodules, with composition given by (3.6); the identities are the algebras themselves seen as bimodules in the usual way. Note that a bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$ is invertible in Alg if and only if it satisfies (3.4) for some bimodule ${}_{\mathcal{B}}Y_{\mathcal{A}}$, so the notion of isomorphism in Alg coincides with Morita equivalence.

This is part of Morita’s theorem [64], see also [4].

Theorem 3.1 *Let \mathcal{A} and \mathcal{B} be unital k -algebras.*

1. *A functor $\mathcal{F} : {}_{\mathcal{B}}\mathfrak{M} \rightarrow {}_{\mathcal{A}}\mathfrak{M}$ is an equivalence of categories if and only if there exists an invertible $(\mathcal{A}, \mathcal{B})$ -bimodule X such that $\mathcal{F} \cong \mathcal{F}_X$.*
2. *A bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$ is invertible if and only if it is finitely generated and projective as a left \mathcal{A} -module and as a right \mathcal{B} -module, and $\mathcal{A} \rightarrow \text{End}_{\mathcal{B}}(X)$ and $\mathcal{B} \rightarrow \text{End}_{\mathcal{A}}(X)$ are algebra isomorphisms.*

Example 3.2 (*Matrix algebras*)

A unital algebra \mathcal{A} is Morita equivalent to the matrix algebra $M_n(\mathcal{A})$, for any $n \geq 1$, through the $(M_n(\mathcal{A}), \mathcal{A})$ -bimodule \mathcal{A}^n .

The following is a geometric example.

Example 3.3 (*Endomorphism bundles*)

Let $\mathcal{A} = C^\infty(M)$ be the algebra of complex-valued functions on a manifold M . The Serre-Swan theorem asserts that any finitely generated projective module over $C^\infty(M)$ can be identified with the space of smooth sections $\Gamma(E)$ of a complex vector bundle $E \rightarrow M$. In fact, $C^\infty(M)$ is Morita equivalent to $\Gamma(\text{End}(E))$ via the $(\Gamma(\text{End}(E)), C^\infty(M))$ -bimodule $\Gamma(E)$. When E is the trivial bundle $\mathbb{C}^n \times M \rightarrow M$, we recover the Morita equivalence of $C^\infty(M)$ and $M_n(C^\infty(M))$ in Example 3.2. The same conclusion holds if \mathcal{A} is the algebra of complex-valued continuous functions on a compact Hausdorff space.

Morita equivalence preserves many algebraic properties besides categories of representations, including ideal structures, cohomology groups and deformation theories [4, 38]. Another important Morita invariant is the center $\mathcal{Z}(\mathcal{A})$ of a unital algebra \mathcal{A} . If X is an invertible $(\mathcal{A}, \mathcal{B})$ -bimodule then, for each $b \in \mathcal{Z}(\mathcal{B})$, there is a unique $a = a(b) \in \mathcal{Z}(\mathcal{A})$ determined by the condition $ax = xb$ for all $x \in X$. In this way, X defines an isomorphism

$$h_X : \mathcal{Z}(\mathcal{A}) \leftarrow \mathcal{Z}(\mathcal{B}), \quad h_X(b) = a(b). \quad (3.7)$$

The group of automorphisms of an object \mathcal{A} in Alg is called its **Picard group**, denoted by $\text{Pic}(\mathcal{A})$. More generally, the invertible morphisms in Alg form a “large” groupoid, called the **Picard groupoid** [9], denoted by Pic . (Here, “large” refers to the fact that the collection of objects in Pic is not a set, though the collection of morphisms between any two of them is.) The set of morphisms from \mathcal{B} to \mathcal{A} are the Morita equivalences; we denote this set by $\text{Pic}(\mathcal{A}, \mathcal{B})$. Of course $\text{Pic}(\mathcal{A}, \mathcal{A}) = \text{Pic}(\mathcal{A})$. The orbit of an object \mathcal{A} in Pic is its Morita equivalence class, while its isotropy $\text{Pic}(\mathcal{A})$ parametrizes the different ways \mathcal{A} can be Morita equivalent to any other object in its orbit. It is clear from this picture that Picard groups of Morita equivalent algebras are isomorphic.

Let us investigate the difference between $\text{Aut}(\mathcal{A})$, the group of ordinary algebra automorphisms of \mathcal{A} , and $\text{Pic}(\mathcal{A})$. Since ordinary automorphisms of \mathcal{A} can be seen as generalized ones, see (3.5), we obtain a group homomorphism

$$j : \text{Aut}(\mathcal{A}) \rightarrow \text{Pic}(\mathcal{A}). \quad (3.8)$$

A simple computation shows that $\ker(j) = \text{InnAut}(\mathcal{A})$, the group of inner automorphisms of \mathcal{A} . So the outer automorphisms $\text{OutAut}(\mathcal{A}) := \text{Aut}(\mathcal{A})/\text{InnAut}(\mathcal{A})$ sit inside $\text{Pic}(\mathcal{A})$.

Exercise

Morita equivalent algebras have isomorphic Picard groups. Do they always have isomorphic groups of outer automorphisms? (Hint: consider the direct sum of two matrix algebras of the same or different sizes.)

On the other hand, (3.7) induces a group homomorphism

$$h : \text{Pic}(\mathcal{A}) \rightarrow \text{Aut}(\mathcal{Z}(\mathcal{A})), \quad (3.9)$$

whose kernel is denoted by $\text{SPic}(\mathcal{A})$, the **static Picard group** of \mathcal{A} .

Remark 3.4 If \mathcal{A} is commutative, then each invertible bimodule induces an automorphism of \mathcal{A} by (3.7), and $\text{SPic}(\mathcal{A})$ consists of those bimodules “fixing” \mathcal{A} , which motivates our terminology. Bimodules in $\text{SPic}(\mathcal{A})$ can also be characterized by having equal left and right module structures, and $\text{SPic}(\mathcal{A})$ is often referred to in the literature as the “commutative” Picard group of \mathcal{A} .

If \mathcal{A} is commutative, then the composition

$$\mathrm{Aut}(\mathcal{A}) \xrightarrow{j} \mathrm{Pic}(\mathcal{A}) \xrightarrow{h} \mathrm{Aut}(\mathcal{A})$$

is the identity. As a result, we can write $\mathrm{Pic}(\mathcal{A})$ as a semi-direct product,

$$\mathrm{Pic}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}) \ltimes \mathrm{SPic}(\mathcal{A}). \quad (3.10)$$

The action of $\mathrm{Aut}(\mathcal{A})$ on $\mathrm{SPic}(\mathcal{A})$ is given by $X \mapsto {}_q X_q$, where the left and right \mathcal{A} -module structures on ${}_q X_q$ are $a \cdot x := q(a)x$ and $x \cdot b := xq(b)$. Although the orbits of *commutative* algebras in Pic are just their isomorphism classes in the ordinary sense, (3.10) illustrates that their isotropy groups in Pic may be bigger than their ordinary automorphism groups. The following is a geometric example.

Example 3.5 (*Picard groups of algebras of functions*)

Let $\mathcal{A} = C^\infty(M)$ be the algebra of smooth complex-valued functions on a manifold M . Using the Serre-Swan identification of smooth complex vector bundles over M with projective modules over \mathcal{A} , one can check that $\mathrm{SPic}(\mathcal{A})$ coincides with $\mathrm{Pic}(M)$, the group of isomorphism classes of complex line bundles on M , which is isomorphic to $H^2(M, \mathbb{Z})$ via the Chern class map. We then have a purely geometric description of $\mathrm{Pic}(\mathcal{A})$ as

$$\mathrm{Pic}(C^\infty(M)) = \mathrm{Diff}(M) \ltimes H^2(M, \mathbb{Z}), \quad (3.11)$$

where the action of $\mathrm{Diff}(M)$ on $H^2(M, \mathbb{Z})$ is given by pull back. In (3.11), we use the identification of algebra automorphisms of \mathcal{A} with diffeomorphisms of M , see e.g. [66].

3.2 Strong Morita equivalence of C^* -algebras

The notion of Morita equivalence of unital algebras has been adapted to several other classes of algebras. An example is the notion of *strong Morita equivalence* of C^* -algebras, introduced by Rieffel in [72, 73].

A C^* -algebra \mathcal{A} is a complex Banach algebra with an involution $*$ such that

$$\|aa^*\| = \|a\|^2, \quad a \in \mathcal{A}.$$

Important examples are the algebra of complex-valued continuous functions on a locally compact Hausdorff space and $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space \mathcal{H} .

The relevant category of modules over a C^* -algebra, to be preserved under strong Morita equivalence, is that of Hilbert spaces on which the C^* -algebra acts through bounded operators. More precisely, for a given C^* -algebra \mathcal{A} , we consider the category $\mathrm{Herm}(\mathcal{A})$ whose objects are pairs (\mathcal{H}, ρ) , where \mathcal{H} is a Hilbert space and $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a *nondegenerate* $*$ -homomorphism of algebras, and morphisms are bounded linear intertwiners. (Here “nondegenerate” means that $\rho(\mathcal{A})h = 0$ implies that $h = 0$, which is always satisfied if \mathcal{A} is unital and ρ preserves the unit.)

Since we are now dealing with more elaborate modules, it is natural that a bimodule giving rise to a functor $\mathrm{Herm}(\mathcal{B}) \rightarrow \mathrm{Herm}(\mathcal{A})$ analogous to (3.2) should be equipped with extra structure. If $(\mathcal{H}, \rho) \in \mathrm{Herm}(\mathcal{B})$ and ${}_A X_B$ is an $(\mathcal{A}, \mathcal{B})$ -bimodule, the key observation is that if X is itself equipped with an inner product $\langle \cdot, \cdot \rangle_B$ with values in \mathcal{B} , then the map ${}_A X_B \otimes_B \mathcal{H} \times {}_A X_B \otimes_B \mathcal{H} \rightarrow \mathbb{C}$ uniquely defined by

$$(x_1 \otimes h_1, x_2 \otimes h_2) \mapsto \langle h_1, \rho(\langle x_1, x_2 \rangle_B) h_2 \rangle \quad (3.12)$$

is an inner product on ${}_A X_B \otimes_B \mathcal{H}$, which we can complete to obtain a Hilbert space \mathcal{H}' . Moreover, the natural \mathcal{A} -action on ${}_A X_B \otimes_B \mathcal{H}$ gives rise to a $*$ -representation $\rho' : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H}')$. These are the main ingredients of Rieffel's induction of representations [72].

More precisely, let X be a right \mathcal{B} -module. Then a \mathcal{B} -valued inner product $\langle \cdot, \cdot \rangle_B$ on X is a \mathbb{C} -sesquilinear pairing $X \times X \rightarrow \mathcal{B}$ (linear in the second argument) such that, for all $x_1, x_2 \in X$ and $b \in \mathcal{B}$, we have

$$\langle x_1, x_2 \rangle_B = \langle x_2, x_1 \rangle_B^*, \quad \langle x_1, x_2 b \rangle_B = \langle x_1, x_2 \rangle_B b, \quad \text{and} \quad \langle x_1, x_1 \rangle_B > 0 \text{ if } x_1 \neq 0.$$

(Inner products on left modules are defined analogously, but linearity is required in the first argument). One can show that $\|x\|_B := \|\langle x, x \rangle_B\|^{1/2}$ is a norm in X . A (right) **Hilbert \mathcal{B} -module** is a (right) \mathcal{B} -module X together with a \mathcal{B} -valued inner product $\langle \cdot, \cdot \rangle_B$ so that X is complete with respect to $\|\cdot\|_B$. Just as for Hilbert spaces, we denote by $\mathfrak{B}_B(X)$ the algebra of endomorphisms of X possessing an adjoint with respect to $\langle \cdot, \cdot \rangle_B$.

Example 3.6 (*Hilbert spaces*)

If $\mathcal{B} = \mathbb{C}$, then Hilbert \mathcal{B} -modules are just ordinary Hilbert spaces. In this case, $\mathfrak{B}_\mathbb{C}(X)$ coincides with the algebra of bounded linear operators on X , see e.g. [71].

Example 3.7 (*Hermitian vector bundles*)

Suppose $\mathcal{B} = C(X)$, the algebra of complex-valued continuous functions on a compact Hausdorff space X . If $E \rightarrow X$ is a complex vector bundle equipped with a hermitian metric h , then $\Gamma(E)$ is a Hilbert \mathcal{B} -module with respect to the $C(X)$ -valued inner product

$$\langle e, f \rangle_B(x) := h_x(e(x), f(x)).$$

To describe the most general Hilbert modules over $C(X)$, one needs Hilbert bundles, which recover Example 3.6 when X is a point.

Example 3.8 (*C^* -algebras*)

Any C^* -algebra \mathcal{B} is a Hilbert \mathcal{B} -module with respect to the inner product $\langle b_1, b_2 \rangle_B = b_1^* b_2$.

As in the case of unital algebras, one can define, for C^* -algebras \mathcal{A} and \mathcal{B} , a “generalized morphism” $\mathcal{A} \leftarrow \mathcal{B}$ as a right Hilbert \mathcal{B} -module X , with inner product $\langle \cdot, \cdot \rangle_B$, together with a *nondegenerate* $*$ -homomorphism $\mathcal{A} \rightarrow \mathfrak{B}_B(X)$. We “compose” ${}_A X_B$ and ${}_B Y_C$ through a more elaborate tensor product: we consider the algebraic tensor product ${}_A X_B \otimes_{\mathbb{C}} {}_B Y_C$, equipped with the semi-positive \mathcal{C} -valued inner product uniquely defined by

$$(x_1 \otimes y_1, x_2 \otimes y_2) \mapsto \langle y_1, \langle x_1, x_2 \rangle_B y_2 \rangle_C. \quad (3.13)$$

The null space of this inner product coincides with the span of elements of the form $xb \otimes y - x \otimes by$ [50], so (3.13) induces a positive-definite \mathcal{C} -valued inner product on ${}_A X_B \otimes_{\mathcal{B}} {}_B Y_C$. The completion of this space with respect to $\|\cdot\|_{\mathcal{C}}$ yields a “generalized morphism” from \mathcal{C} to \mathcal{A} denoted by ${}_A \widehat{X_B \otimes_{\mathcal{B}} Y_C}$, called the **Rieffel tensor product** of ${}_A X_B$ and ${}_B Y_C$.

An isomorphism between “generalized morphisms” is a bimodule isomorphism preserving inner products. Just as ordinary tensor products, Rieffel tensor products are associative up to natural isomorphisms. So one can define a category \mathbf{C}^* whose objects are C^* -algebras and whose morphisms are isomorphism classes of “generalized morphisms”, with composition given by Rieffel tensor product; the identities are the algebras themselves, regarded as bimodules in the usual way, and with the inner product of Example 3.8.

Two C^* -algebras are **strongly Morita equivalent** if they are isomorphic in C^* . As in the case of unital algebras, isomorphic C^* -algebras are necessarily strongly Morita equivalent.

Remark 3.9 (*Equivalence bimodules*)

The definition of strong Morita equivalence as isomorphism in C^* coincides with Rieffel's original definition in terms of equivalence bimodules (also called imprimitivity bimodules) [72, 73]. In fact, any “generalized morphism” ${}_A X_B$ which is invertible in C^* can be endowed with an \mathcal{A} -valued inner product, compatible with its \mathcal{B} -valued inner product in the appropriate way, making it into an equivalence bimodule, see [52] and references therein. Conversely, any equivalence bimodule is automatically invertible in C^* .

Example 3.10 (*Compact operators*)

A Hilbert space \mathcal{H} , seen as a bimodule for \mathbb{C} and the C^* -algebra $\mathcal{K}(\mathcal{H})$ of compact operators on \mathcal{H} , defines a strong Morita equivalence.

Example 3.11 (*Endomorphism bundles*)

Analogously to Example 3.3, a hermitian vector bundle $E \rightarrow X$, where X is a compact Hausdorff space, defines a strong Morita equivalence between $\Gamma(\text{End}(E))$ and $C(X)$.

Any “generalized morphism” ${}_A X_B$ in C^* defines a functor

$$\mathcal{F}_X : \text{Herm}(\mathcal{B}) \rightarrow \text{Herm}(\mathcal{A}),$$

similar to (3.2), but with Rieffel's tensor product replacing the ordinary one, i.e., on objects,

$$\mathcal{F}_X(\mathcal{H}) := {}_A X_B \hat{\otimes}_B \mathcal{H}. \quad (3.14)$$

Such a functor is called **Rieffel induction of representations** [72]. It follows that strongly Morita equivalent C^* -algebras have equivalent categories of representations, although, in this setting, the converse is not true [73] (see [11] for a different approach where a converse does hold).

Remark 3.12 (*Strong vs. ring-theoretic Morita equivalence*)

By regarding unital C^* -algebras simply as unital algebras over \mathbb{C} , one can compare strong and ring-theoretic Morita equivalences. It turns out that two unital C^* -algebras are strongly Morita equivalent if and only if they are Morita equivalent as unital \mathbb{C} -algebras [6]. However, the Picard groups associated to each notion are different in general, see [18]. In terms of Picard groupoids, this means that, over unital C^* -algebras, the Picard groupoids associated with ring-theoretic and strong Morita equivalences have the same orbits, but generally different isotropy groups.

A study of Picard groups associated with strong Morita equivalence, analogous to the discussion in Section 3.1, can be found in [12].

3.3 Morita equivalence of deformed algebras

Let (P, Π) be a Poisson manifold and $C^\infty(P)$ be its algebra of smooth complex-valued functions. The general idea of a **deformation quantization** of P “in the direction” of Π is that of a family \star_\hbar of associative algebra structures on $C^\infty(P)$ satisfying the following two conditions:

$$i.) \quad f \star_\hbar g = f \cdot g + O(\hbar);$$

ii.) $\frac{1}{i\hbar}(f \star_{\hbar} g - g \star_{\hbar} f) \longrightarrow \{f, g\}$, when $\hbar \rightarrow 0$.

There are several versions of deformation quantization. We will consider

1. **Formal deformation quantization** [5]: In this case, \star_{\hbar} is an associative product on $C^{\infty}(P)[[\hbar]]$, the space of formal power series with coefficients in $C^{\infty}(P)$. Here \hbar is a formal parameter, and the “limit” in ii.) above is defined simply by setting \hbar to 0. A formal deformation quantization is also called a **star product**. The contribution by Cattaneo and Indelicato [23] to this volume contains a thorough exposition of the theory of star products and its history.
2. **Rieffel’s strict deformation quantization** [75]: In this setting, one starts with a dense Poisson subalgebra of $C_{\infty}(P)$, the C^* -algebra of continuous functions on P vanishing at infinity, and considers families of associative products \star_{\hbar} on it, defined along with norms and involutions such that the completions form a continuous field of C^* -algebras. The parameter \hbar belongs to a closed subset of \mathbb{R} having 0 as a non-isolated point, and one can make analytical sense of the limit in ii.) above. Variations of Rieffel’s notion of deformation quantization are discussed in [51].

Intuitively, one should regard a deformation quantization \star_{\hbar} as a path in the “space of associative algebra structures” on $C^{\infty}(P)$ for which the Poisson structure Π is the “tangent vector” at $\hbar = 0$. From this perspective, a direct relationship between deformation quantization and Poisson geometry is more likely in the formal case.

A natural question is when two algebras obtained by deformation quantization are Morita equivalent. In the framework of formal deformation quantization, the first observation is that if two deformation quantizations $(C^{\infty}(P_1, \Pi_1)[[\hbar]], \star_{\hbar}^1)$ and $(C^{\infty}(P_2, \Pi_2)[[\hbar]], \star_{\hbar}^2)$ are Morita equivalent (as unital algebras over $\mathbb{C}[[\hbar]]$), then the underlying Poisson manifolds are isomorphic. So we can restrict ourselves to a fixed Poisson manifold. The following result is proven in [13, 17]:

Theorem 3.13 *Let P be symplectic. If $\text{Pic}(P) \cong H^2(P, \mathbb{Z})$ has no torsion, then it acts freely on the set of equivalence classes of star products on P , and two star products are Morita equivalent if and only if their classes lie in the same $H^2(P, \mathbb{Z})$ -orbit, up to symplectomorphism.*

Recall that two star products \star_{\hbar}^1 and \star_{\hbar}^2 are **equivalent** if there exists a family of differential operators $T_r : C^{\infty}(P) \rightarrow C^{\infty}(P)$, $r = 1, 2, \dots$, so that $T = \text{Id} + \sum_{r=1}^{\infty} T_r \hbar^r$ is an algebra isomorphism

$$(C^{\infty}(P)[[\hbar]], \star_{\hbar}^1) \xrightarrow{\sim} (C^{\infty}(P)[[\hbar]], \star_{\hbar}^2).$$

Equivalence classes of star products on a symplectic manifold are parametrized by elements in

$$\frac{1}{i\hbar}[\omega] + H_{dR}^2(P)[[\hbar]], \quad (3.15)$$

where ω is the symplectic form on P and $H_{dR}^2(P)$ is the second de Rham cohomology group of P with complex coefficients [5, 36, 67], called **characteristic classes**. As shown in [17], the $\text{Pic}(P)$ -action of Theorem 3.13 is explicitly given in terms of these classes by

$$[\omega_{\hbar}] \mapsto [\omega_{\hbar}] + 2\pi i c_1(L), \quad (3.16)$$

where $[\omega_{\hbar}]$ is an element in (3.15) and $c_1(L)$ is the image of the Chern class of the line bundle L in $H_{dR}^2(P)$.

Remark 3.14 A version of Theorem 3.13 holds for arbitrary Poisson manifolds (P, Π) , see [13, 46]. In this general setting, equivalence classes of star products are parametrized by classes of formal Poisson bivectors $\Pi_{\hbar} = \Pi + \hbar \Pi_1 + \dots$ (see [48] or the exposition in [23]), and the $\text{Pic}(P)$ -action on them, classifying Morita equivalent deformation quantizations of P , is via gauge transformations (see Section 4.8).

In the framework of strict deformation quantization and the special case of tori, a classification result for Morita equivalence was obtained by Rieffel and Schwarz in [76] (see also [54, 81]). Let us consider $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ equipped with a constant Poisson structure, represented by a skew-symmetric real matrix Π : if $(\theta_1, \dots, \theta_n)$ are coordinates on T^n , then

$$\Pi_{ij} = \{\theta_i, \theta_j\}.$$

Via the Fourier transform, one can identify the algebra $C^\infty(\mathbb{T}^n)$ with the space $\mathcal{S}(\mathbb{Z}^n)$ of complex-valued functions on \mathbb{Z}^n with rapid decay at infinity. Under this identification, the pointwise product of functions becomes the convolution on $\mathcal{S}(\mathbb{Z}^n)$,

$$\widehat{f} * \widehat{g}(n) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(n) \widehat{g}(n - k),$$

$\widehat{f}, \widehat{g} \in \mathcal{S}(\mathbb{Z}^n)$. One can now use the matrix Π to “twist” the convolution and define a new product

$$\widehat{f} *_\hbar \widehat{g}(n) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(n) \widehat{g}(n - k) e^{-\pi i \hbar \Pi(k, n-k)} \quad (3.17)$$

on $\mathcal{S}(\mathbb{Z}^n)$, which can be pulled back to a new product in $C^\infty(\mathbb{T}^n)$. Here \hbar is a real parameter. If we set $\hbar = 1$, this defines the algebra \mathcal{A}_Π^∞ , which can be thought of as the “algebra of smooth functions on the quantum torus \mathbb{T}_Π^n ”. A suitable completion of \mathcal{A}_Π^∞ defines a C^* -algebra \mathcal{A}_Π , which is then thought of as the “algebra of continuous functions on \mathbb{T}_Π^n ”. (Note that, with $\hbar = 1$, we are no longer really considering a deformation.)

Exercise

Show that 1 is a unit for \mathcal{A}_Π . Let $u_j = e^{2\pi i \theta_j}$. Show that $u_j *_1 \bar{u}_j = \bar{u}_j *_1 u_j = 1$ and

$$u_j *_1 u_k = e^{2\pi i \Pi_{jk}} u_k *_1 u_j. \quad (3.18)$$

The algebra \mathcal{A}_Π can be alternatively described as the universal C^* -algebra generated by n unitary elements u_1, \dots, u_n subject to the commutation relations (3.18).

In this context, the question to be addressed is when skew-symmetric matrices Π and Π' correspond to Morita equivalent C^* -algebras \mathcal{A}_Π and $\mathcal{A}_{\Pi'}$. Let $O(n, n|\mathbb{R})$ be the group of linear automorphisms of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ preserving the inner product (2.6). One can identify elements of $O(n, n|\mathbb{R})$ with matrices

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $n \times n$ matrices satisfying

$$A^t C + C^t A = 0 = B^t D + D^t B, \quad \text{and} \quad A^t D + C^t B = 1.$$

The group $O(n, n|\mathbb{R})$ “acts” on the space of all $n \times n$ skew-symmetric matrices by

$$\Pi \mapsto g \cdot \Pi := (A\Pi + B)(C\Pi + D)^{-1}. \quad (3.19)$$

Note that this is not an honest action, since the formula above only makes sense when $(C\Pi + D)$ is invertible.

Let $SO(n, n|\mathbb{Z})$ be the subgroup of $O(n, n|\mathbb{R})$ consisting of matrices with integer coefficients and determinant 1. The main result of [76], as improved in [54, 81], is

Theorem 3.15 *If Π is a skew-symmetric matrix, $g \in SO(n, n|\mathbb{Z})$ and $g \cdot \Pi$ is defined, then \mathcal{A}_Π and $\mathcal{A}_{g \cdot \Pi}$ are strongly Morita equivalent.*

Remark 3.16 (*Converse results*)

The converse of Theorem 3.15 holds for $n = 2$ [74], but not in general. In fact, for $n = 3$, one can find Π and Π' , *not* in the same $SO(n, n|\mathbb{Z})$ -orbit, for which \mathcal{A}_Π and $\mathcal{A}_{\Pi'}$ are isomorphic (hence Morita equivalent) [76].

On the other hand, for *smooth* quantum tori, Theorem 3.15 and its converse hold with respect to a refined notion of Morita equivalence, called “complete Morita equivalence” [77], in which bimodules carry connections of constant curvature.

For the algebraic Morita equivalence of smooth quantum tori, see [35].

Remark 3.17 (*Dirac structures and quantum tori*)

In [76], the original version of Theorem 3.15 was proven under an additional hypothesis. Rieffel and Schwarz consider three types of generators of $SO(n, n|\mathbb{Z})$, and prove that their action preserves Morita equivalence. In order to show that \mathcal{A}_Π and $\mathcal{A}_{g\Pi}$ are Morita equivalent for an arbitrary $g \in SO(n, n|\mathbb{Z})$ (for which $g\Pi$ is defined), they need to assume that g can be written as a product of generators $g_r \cdots g_1$ in such a way that each of the products $g_k \cdots g_1 \Pi$ is defined. The result in Theorem 3.15, without this assumption, is conjectured in [76], and it was proven by Li in [54].

A geometric way to circumvent the difficulties in the Rieffel-Schwarz proof, in which Dirac structures play a central role, appears in [81]. The key point is the observation that, even if $g \cdot \Pi$ is not defined as a skew-symmetric matrix, it is still a Dirac structure on \mathbb{T}^n . The authors develop a way to quantize constant Dirac structures on \mathbb{T}^n by attaching to each one of them a Morita equivalence class of quantum tori. They extend the $SO(n, n|\mathbb{Z})$ action to Dirac structures and prove that the Morita equivalence classes of the corresponding quantum tori is unchanged under the action.

4 Geometric Morita equivalence

In this section, we introduce a purely geometric notion of Morita equivalence of Poisson manifolds. This notion leads inevitably to the consideration of Morita equivalence of symplectic groupoids, so we will make a digression into the Morita theory of general Lie groups and groupoids. We end the section with a discussion of gauge equivalence, a geometric equivalence which is close to Morita equivalence, but is also related to the algebraic Morita equivalence of star products, as discussed in Section 3.3.

4.1 Representations and tensor product

In order to define Morita equivalence in Poisson geometry, we need notions of “representations” of (or “modules” over) Poisson manifolds as well as their tensor products.

As we saw in Example 2.24, symplectic manifolds are in some sense “irreducible” among Poisson manifolds. If one thinks of Poisson manifolds as algebras, then symplectic manifolds

could be thought of as “matrix algebras”. Following this analogy, a representation of a Poisson manifold P should be a symplectic manifold S together with a Poisson map $J : S \rightarrow P$ which is complete. At the level of functions, we have a “representation” of $C^\infty(P)$ by $J^* : C^\infty(P) \rightarrow C^\infty(S)$. This notion of representation is also suggested by the theory of geometric quantization, in which symplectic manifolds become “vector spaces” on which their Poisson algebras “act asymptotically”.

More precisely, we define a left [right] **P -module** to be a complete [anti-] symplectic realization $J : S \rightarrow P$. Our first example illustrates how modules over Lie-Poisson manifolds are related to hamiltonian actions.

Example 4.1 (*Modules over \mathfrak{g}^* and hamiltonian actions*)

Let (S, Π_S) be a symplectic Poisson manifold, \mathfrak{g} be a Lie algebra, and suppose that $J : S \rightarrow \mathfrak{g}^*$ is a symplectic realization of \mathfrak{g}^* . The map

$$\mathfrak{g} \rightarrow \mathcal{X}(S), \quad v \mapsto \tilde{\Pi}_S(dJ_v), \quad (4.1)$$

where $J_v(x) := \langle J(x), v \rangle$, defines a \mathfrak{g} -action on S by hamiltonian vector fields for which J is the momentum map. On the other hand, the momentum map $J : S \rightarrow \mathfrak{g}^*$ for a hamiltonian \mathfrak{g} -action on S is a Poisson map, so we have a one-to-one correspondence between symplectic realizations of \mathfrak{g}^* and hamiltonian \mathfrak{g} -manifolds.

A symplectic realization $J : S \rightarrow \mathfrak{g}^*$ is complete if and only if the associated infinitesimal hamiltonian action is by complete vector fields, in which case it can be integrated to a hamiltonian G -action, where G is the connected and simply-connected Lie group having \mathfrak{g} as its Lie algebra. So \mathfrak{g}^* -modules are just the same thing as hamiltonian G -manifolds.

Remark 4.2 (*More general modules over \mathfrak{g}^**)

The one-to-one correspondence in Example 4.1 extends to one between Poisson maps into \mathfrak{g}^* (from any Poisson manifold, not necessarily symplectic) and hamiltonian \mathfrak{g} -actions on Poisson manifolds, or, similarly, between complete Poisson maps into \mathfrak{g}^* and Poisson manifolds carrying hamiltonian G -actions. This indicates that it may be useful to regard *arbitrary* (complete) Poisson maps as modules over Poisson manifolds; we will say more about this in Remarks 4.18 and 4.24.

We now define a tensor product operation on modules over a Poisson manifold. Let $J : S \rightarrow P$ be a right P -module, and let $J' : S' \rightarrow P$ be a left P -module. Just as, in algebra, we can think of the tensor product over \mathcal{A} of a left module X and a right module Y as a quotient of their tensor product over the ground ring k , so in Poisson geometry we can define the tensor product of S and S' to be a “symplectic quotient” of $S \times S'$. Namely, the fibre product

$$S \times_{(J, J')} S' = \{(x, y) \in S \times S' \mid J(x) = J'(y)\} \quad (4.2)$$

is the inverse image of the diagonal under the Poisson map $(J, J') : S \times S' \rightarrow P \times \overline{P}$, hence, whenever it is smooth, it is a coisotropic submanifold. (Here \overline{P} denotes P equipped with its Poisson structure multiplied by -1.) Let us assume then, that the fibre product is smooth; this is the case, for example, if either J or J' is a surjective submersion. Then we may define the **tensor product** $S * S'$ over P to be the quotient of this fibre product by its characteristic foliation. In general, even if the fibre product is smooth, $S * S'$ is still not a smooth manifold, but just a quotient of a manifold by a foliation. We will have to deal with this problem later, see Remark 4.40. But when the characteristic foliation is simple, $S * S'$ is a symplectic manifold.

We may write $S *_P S'$ instead of $S * S'$ to identify the Poisson manifold over which we are taking the tensor product.

If one is given two left modules (one could do the same for right modules, of course), one can apply the tensor product construction by changing the “handedness” of one of them. Thus, if S and S' are left P -modules, then $\overline{S'}$ is a right module, and we can form the tensor product $\overline{S'} * S$. We call this the **classical intertwiner space** [94, 95] of S and S' and denote it by $\text{Hom}(S, S')$. The name and notation come from the case of modules over an algebra, where the tensor product $Y^* \otimes X$ is naturally isomorphic to the space of module homomorphisms from Y to X when these modules are “finite dimensional”. When the algebra is a group algebra, the modules are representations of the group, and the module homomorphisms are known as intertwining operators.

Example 4.3 (*Symplectic reduction*)

Let $J : S \rightarrow \mathfrak{g}^*$ be the momentum map for a hamiltonian action of a connected Lie group G on a symplectic manifold S . Let $S' = \mathcal{O}_\mu$ be the coadjoint orbit through $\mu \in \mathfrak{g}^*$, equipped with the symplectic structure induced by the Lie-Poisson structure on \mathfrak{g}^* , and let $\iota : \mathcal{O}_\mu \hookrightarrow \mathfrak{g}^*$ be the inclusion, which is a Poisson map. Then the classical intertwiner space $\text{Hom}(S, \mathcal{O}_\mu)$ is equal to $J^{-1}(\mathcal{O}_\mu)/G \cong J^{-1}(\mu)/G_\mu$, i.e., the symplectic reduction of S at the momentum value μ .

A (P_1, P_2) -**bimodule** is a symplectic manifold S and a pair of maps $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ making S into a left P_1 -module and a right P_2 -module and satisfying the “commuting actions” condition:

$$\{J_1^* C^\infty(P_1), J_2^* C^\infty(P_2)\} = 0. \quad (4.3)$$

(Such geometric bimodules, without the completeness assumption, are called **dual pairs** in [87].) An **isomorphism** of bimodules is a symplectomorphism commuting with the Poisson maps.

Given bimodules $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ and $P_2 \xleftarrow{J'_2} S' \xrightarrow{J'_3} P_3$, we may form the tensor product $S *_P S'$, and it is easily seen that this tensor product, whenever it is smooth, becomes a (P_1, P_3) -bimodule [94, 52]. We think of this tensor product as the **composition** of S and S' .

Remark 4.4 (*Modules as bimodules and geometric Rieffel Induction*)

For any left P_2 -module S' , there is an associated bimodule $P_2 \leftarrow S' \rightarrow \text{pt}$, where pt is just a point. Given a bimodule $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$, we can form its tensor product with $P_2 \leftarrow S' \rightarrow \text{pt}$ to get a (P_1, pt) -bimodule. In this way, the (P_1, P_2) -bimodule “acts” on P_2 -modules to give P_1 -modules. This is the geometric analogue of the functors (3.2) and (3.14), for unital and C^* -algebras, respectively.

Example 4.5 Following Example 4.3, suppose that the orbit space S/G is smooth, in which case it is a Poisson manifold in a natural way. Consider the bimodules $S/G \leftarrow S \xrightarrow{J} \overline{\mathfrak{g}^*}$ and $\overline{\mathfrak{g}^*} \xleftarrow{\iota} \overline{\mathcal{O}_\mu} \rightarrow \text{pt}$. Their tensor product is the $(S/G, \text{pt})$ -bimodule $S/G \leftarrow S * \overline{\mathcal{O}_\mu} \rightarrow \text{pt}$, where the map on the left is the inclusion of the symplectic reduced space $S * \overline{\mathcal{O}_\mu} = \text{Hom}(S, \mathcal{O})$ as a symplectic leaf of S/G .

Following the analogy with algebras, it is natural to think of isomorphism classes of bimodules as generalized morphisms of Poisson manifolds. The extra technical difficulty in this geometric context is that tensor products do not always result in smooth spaces. So one needs a suitable notion of “regular bimodules”, satisfying extra regularity conditions to guarantee that their tensor products are smooth and again “regular”, see [19, 52], or an appropriate notion of bimodule modeled on “singular” spaces. We will come back to these topics in Section 4.7.

4.2 Symplectic groupoids

In order to regard geometric bimodules over Poisson manifolds as morphisms in a category, one needs to identify the bimodules which serve as identities, i.e., those satisfying

$$S * S' \cong S' \quad \text{and} \quad S'' * S \cong S''$$

for any other bimodules S' and S'' . As we saw in Section 3.1, in the case of unital algebras, the identity bimodule of an object \mathcal{A} in \mathbf{Alg} is just \mathcal{A} itself, regarded as an $(\mathcal{A}, \mathcal{A})$ -bimodule in the usual way. This idea cannot work for Poisson manifolds, since they are generally not symplectic, and because we do not have commuting left and right actions of P on itself. Instead, it is the symplectic groupoids [88] which serve as such “identity bimodules” for Poisson manifolds, see [52]. If $P \xleftarrow{t} \mathcal{G} \xrightarrow{s} P$ is an identity bimodule for a Poisson manifold P , then there exists, in particular, a symplectomorphism $\mathcal{G} * \mathcal{G} \rightarrow \mathcal{G}$, and the composition

$$\mathcal{G} \times_{(s,t)} \mathcal{G} \rightarrow \mathcal{G} * \mathcal{G} \xrightarrow{\sim} \mathcal{G}$$

defines a map $m : \mathcal{G} \times_{(s,t)} \mathcal{G} \rightarrow \mathcal{G}$ which turns out to be a groupoid multiplication³, compatible with the symplectic form on \mathcal{G} in the sense that $\text{graph}(m) \subseteq \mathcal{G} \times \overline{\mathcal{G}} \times \overline{\mathcal{G}}$ is a *lagrangian* submanifold. If $p_i : \mathcal{G} \times_{(s,t)} \mathcal{G} \rightarrow \mathcal{G}$, $i = 1, 2$, are the natural projections, then the compatibility between m and ω is equivalent to the condition

$$m^* \omega = p_1^* \omega + p_2^* \omega. \quad (4.4)$$

A 2-form ω satisfying (4.4) is called **multiplicative** (note that if ω were a function, (4.4) would mean that $\omega(gh) = \omega(g) + \omega(h)$), and a groupoid equipped with a multiplicative symplectic form is called a **symplectic groupoid**.

If (\mathcal{G}, ω) is a symplectic groupoid over a manifold P , then the following important properties follow from the compatibility condition (4.4), see [26]:

- i) The unit section $P \hookrightarrow \mathcal{G}$ is lagrangian;
- ii) The inversion map $\mathcal{G} \rightarrow \mathcal{G}$ is an anti-symplectic involution;
- iii) The fibres of the target and source maps, $t, s : \mathcal{G} \rightarrow P$, are the symplectic orthogonal of one another;
- iv) At each point of \mathcal{G} , $\ker(Ts) = \{X_{t^*f} \mid f \in C^\infty(P)\}$ and $\ker(Tt) = \{X_{s^*f} \mid f \in C^\infty(P)\}$;
- v) P carries a unique Poisson structure such that the target map t is a Poisson map (and the source map s is anti-Poisson).

A Poisson manifold (P, Π) is called **integrable** if there exists a symplectic groupoid (\mathcal{G}, ω) over P which induces Π in the sense of v), and we refer to \mathcal{G} as an **integration** of P . As we will discuss later, not every Poisson manifold is integrable in this sense, see [30, 88]. But if P is integrable, then there exists a symplectic groupoid integrating it which has simply-connected (i.e., connected with trivial fundamental group) source fibres [57], and this groupoid is unique up to isomorphism.

³For expositions on groupoids, we refer to [20, 62, 63]; we adopt the convention that, on a Lie groupoid \mathcal{G} over P , with source s and target t , the multiplication is defined on $\{(g, h) \in \mathcal{G} \times \mathcal{G}, s(g) = t(h)\}$, and we identify the Lie algebroid $A(\mathcal{G})$ with $\ker(Ts)|_P$, and Tt is the anchor map. The bracket on the Lie algebroid comes from identification with *right*-invariant vector fields, which is counter to a convention often used for Lie groups.

Remark 4.6 (*Integrability and complete symplectic realizations*)

If (\mathcal{G}, ω) is an integration of (P, Π) , then the target map $t : \mathcal{G} \rightarrow P$ is a Poisson submersion which is always complete. On the other hand, as proven in [30], if a Poisson manifold P admits a complete symplectic realization $S \rightarrow P$ which is a submersion, then P must be integrable.

Remark 4.7 (*The Lie algebroid of a Poisson manifold*)

All the integrations of a Poisson manifold (P, Π) have (up to natural isomorphism) the same Lie algebroid. It is T^*P , with a Lie algebroid structure with anchor $\tilde{\Pi} : T^*P \rightarrow TP$, and Lie bracket on $\Gamma(T^*P) = \Omega^1(P)$ defined by

$$[\alpha, \beta] := \mathcal{L}_{\tilde{\Pi}(\alpha)}(\beta) - \mathcal{L}_{\tilde{\Pi}(\beta)}(\alpha) - d\Pi(\alpha, \beta). \quad (4.5)$$

Note that (4.5) is uniquely characterized by $[df, dg] = d\{f, g\}$ and the Leibniz identity. Following Remark 2.18, we know that $L_\Pi = \text{graph}(\tilde{\Pi})$ also carries a Lie algebroid structure, induced by the Courant bracket. The natural projection $\text{pr}_2 : TP \oplus T^*P \rightarrow T^*P$ restricts to a vector bundle isomorphism $L_\Pi \rightarrow T^*P$ which defines an isomorphism of Lie algebroids.

On the other hand, if (\mathcal{G}, ω) is a symplectic groupoid integrating (P, Π) , then the bundle isomorphism

$$\ker(Ts)|_P \longrightarrow T^*P, \quad \xi \mapsto i_\xi \omega|_{TP} \quad (4.6)$$

induces an isomorphism of Lie algebroids $A(\mathcal{G}) \xrightarrow{\sim} T^*P$, where $A(\mathcal{G})$ is the Lie algebroid of \mathcal{G} , so the symplectic groupoid \mathcal{G} integrates T^*P in the sense of Lie algebroids. It follows from (4.6) that $\dim(\mathcal{G}) = 2 \dim(P)$.

In the work of Cattaneo and Felder [22], symplectic groupoids arise as reduced phase spaces of Poisson sigma models. This means that one begins with the space of paths on T^*P , which has a natural symplectic structure, restricts to a certain submanifold of “admissible” paths, and forms the symplectic groupoid $\mathcal{G}(P)$ as a quotient of this submanifold by a foliation. This can also be described as an infinite-dimensional symplectic reduction. The resulting space is a groupoid but may not be a manifold. When it is a manifold, it is a the source-simply-connected symplectic groupoid of P . When $\mathcal{G}(P)$ is not a manifold, as the leaf space of a foliation, it can be considered as a differentiable stack, and even as a symplectic stack. In the world of stacks [59], it is again a smooth groupoid; we will call it an **S-groupoid**. The first steps of this program have been carried out by Tseng and Zhu [82]. (See [92] for an exposition, as well as Remark 4.40 below.)

This construction of symplectic groupoids has been extended to general Lie algebroids, see [29, 78]. Crainic and Fernandes [29] describe explicitly the obstructions to the integrability of Lie algebroids and, in [30], identify these obstructions for the case of Poisson manifolds and symplectic groupoids. Integration by S-groupoids is done in [82].

The next three examples illustrate simple yet important classes of integrable Poisson manifolds and their symplectic groupoids.

Example 4.8 (*Symplectic manifolds*)

If (P, ω) is a symplectic manifold, then the pair groupoid $P \times P$ equipped with the symplectic form $\omega \times (-\omega)$ is a symplectic groupoid integrating P . In order to obtain a source-simply-connected integration, one should consider the fundamental groupoid $\pi(P)$, with symplectic structure given by the pull-back of the symplectic form on $P \times \overline{P}$ by the covering map $\pi(P) \rightarrow P \times \overline{P}$.

Example 4.9 (*Zero Poisson brackets*)

If (P, Π) is a Poisson manifold with $\Pi = 0$, then $\mathcal{G}(P) = T^*P$. In this case, the source and target maps coincide with the projection $T^*P \rightarrow P$, and the multiplication on T^*P is given by fibrewise addition. There are, however, other symplectic groupoids integrating P , which may not have connected or simply-connected source fibres. For example, if T^*P admits a basis of closed 1-forms, we may divide the fibres of T^*P by the lattice generated by these forms to obtain a groupoid whose source and target fibres are tori. Or, if P is just a point, any discrete group is a symplectic groupoid for P . We refer to [19] for more details.

Example 4.10 (*Lie-Poisson structures*)

Let $P = \mathfrak{g}^*$ be the dual of a Lie algebra \mathfrak{g} , equipped with its Lie-Poisson structure, and let G be a Lie group with Lie algebra \mathfrak{g} . The transformation groupoid $G \ltimes \mathfrak{g}^*$ with respect to the coadjoint action, equipped with the symplectic form obtained from the identification $G \times \mathfrak{g}^* \cong T^*G$ by right translation, is a symplectic groupoid integrating \mathfrak{g}^* . This symplectic groupoid is source-simply-connected just when G is a (connected) simply-connected Lie group.

Remark 4.11 (*Lie's third theorem*)

Let \mathfrak{g} be a Lie algebra. Example 4.10 shows that integrating \mathfrak{g} , in the usual sense of finding a Lie group G with Lie algebra \mathfrak{g} , yields an integration of the Lie-Poisson structure of \mathfrak{g}^* . On the other hand, one can use the integration of the Lie-Poisson structure of \mathfrak{g}^* to construct a Lie group integrating \mathfrak{g} . Indeed, if \mathcal{G} is a symplectic groupoid integrating \mathfrak{g}^* , then the map

$$\mathfrak{g} \rightarrow \mathcal{X}(\mathcal{G}), \quad v \mapsto X_{t^*v}$$

is a faithful representation of \mathfrak{g} by vector fields on \mathcal{G} . Here $t : \mathcal{G} \rightarrow \mathfrak{g}^*$ is the target map, and we regard $v \in \mathfrak{g}$ as a linear function on \mathfrak{g}^* . We then use the flows of these vector fields to define a (local) Lie group integrating \mathfrak{g} . If we fix $x \in \mathcal{G}$, the “identity” of the local Lie group, so that $t(x) = 0$, then the Lie group sits in \mathcal{G} as a lagrangian subgroupoid. So the two “integrations” are the same.

The idea of using a symplectic realization of \mathfrak{g}^* to find a Lie group integrating \mathfrak{g} goes back to Lie's original proof of “Lie's third theorem.” A regular point of \mathfrak{g}^* has a neighborhood U with coordinates $(q_1, \dots, q_k, p_1, \dots, p_k, e_1, \dots, e_l)$ such that the Lie-Poisson structure can be written as

$$\sum_{i=1}^k \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

(see Section 2.4). The map $\mathfrak{g} \rightarrow \mathcal{X}(U)$, $v \mapsto -X_v$ is a Lie algebra homomorphism, but not faithful in general. It suffices, though, to add l new coordinates (f_1, \dots, f_l) and consider the local symplectic realization $U \times \mathbb{R}^l \rightarrow U$, with symplectic Poisson structure

$$\Pi' = \sum_{i=1}^k \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^l \frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial f_i}.$$

The map $\mathfrak{g} \rightarrow \mathcal{X}(U \times \mathbb{R}^l)$, $v \mapsto -X'_v := \tilde{\Pi}'(v)$, is now a faithful Lie algebra homomorphism. Once again, we can use the flows of the hamiltonian vector fields of the coordinates on \mathfrak{g} to construct a local Lie group.

More generally, if \mathcal{G} is a Lie groupoid and A is its Lie algebroid, then $T^*\mathcal{G}$ is naturally a symplectic groupoid over A^* , see [26]. The induced Poisson structure on A^* is a generalization

of a Lie-Poisson structure. Conversely, if A is an integrable Lie algebroid, then $\mathcal{G}(A)$, its source-simply-connected integration, can be constructed as a lagrangian subgroupoid of the symplectic groupoid $\mathcal{G}(A^*)$ integrating A^* [21].

The following is an example of a non-integrable Poisson structure.

Example 4.12 (*Nonintegrable Poisson structure*)

Let $P = S^2 \times \mathbb{R}$. Let Π_{S^2} be the natural symplectic structure on S^2 . Then the product Poisson structure on P , $\Pi_{S^2} \times \{0\}$ is integrable. But if we multiply this Poisson structure by $(1 + t^2)$, $t \in \mathbb{R}$ (or use any other nonconstant function which has a critical point), then the resulting Poisson structure $(1 + t^2)(\Pi_{S^2} \times \{0\})$ is *not* integrable [30, 88]. In this case the symplectic S-groupoid $\mathcal{G}(P)$ is not a manifold.

We will have more to say about this example in Section 4.7.

Remark 4.13 (*Twisted presymplectic groupoids*)

Let \mathcal{G} be a Lie groupoid over a manifold P . For each $k > 0$, let \mathcal{G}_k be the manifold of composable sequences of k -arrows,

$$\mathcal{G}_k := \mathcal{G} \times_{(s,t)} \mathcal{G} \times_{(s,t)} \cdots \times_{(s,t)} \mathcal{G}, \quad (k \text{ times})$$

and set $\mathcal{G}_0 = P$. The sequence of manifolds \mathcal{G}_k , together with the natural maps $\partial_i : \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$, $i = 0, \dots, k$,

$$\partial_i(g_1, \dots, g_k) = \begin{cases} (g_2, \dots, g_k), & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_k), & \text{if } 0 < i < k \\ (g_1, \dots, g_{k-1}) & \text{if } i = k. \end{cases}$$

defines a simplicial manifold \mathcal{G}_\bullet . The **bar-de Rham complex** of \mathcal{G} is the total complex of the double complex $\Omega^\bullet(\mathcal{G}_\bullet)$, where the boundary maps are $d : \Omega^q(\mathcal{G}_k) \rightarrow \Omega^{q+1}(\mathcal{G}_k)$, the usual de Rham differential, and $\partial : \Omega^q(\mathcal{G}_k) \rightarrow \Omega^q(\mathcal{G}_{k+1})$, the alternating sum of the pull-back of the $k+1$ maps $\mathcal{G}_k \rightarrow \mathcal{G}_{k+1}$, as in group cohomology. For example, if $\omega \in \Omega^2(\mathcal{G})$, then

$$\partial\omega = p_1^*\omega - m^*\omega + p_2^*\omega.$$

(As before, m is the groupoid multiplication, and $p_i : \mathcal{G}_2 \rightarrow \mathcal{G}$, $i = 1, 2$, are the natural projections.) It follows that a 2-form ω is a 3-cocycle in the total complex if and only if it is multiplicative and closed; in particular, a symplectic groupoid can be defined as a Lie groupoid \mathcal{G} together with a nondegenerate 2-form ω which is a 3-cocycle.

More generally, one can consider 3-cochains which are sums $\omega + \phi$, where $\omega \in \Omega^2(\mathcal{G})$ and $\phi \in \Omega^3(P)$. In this case, the coboundary condition is that $d\phi = 0$, ω is multiplicative, and

$$d\omega = s^*\phi - t^*\phi.$$

A groupoid \mathcal{G} together with a 3-cocycle (ω, ϕ) such that ω is nondegenerate is called a **ϕ -twisted symplectic groupoid** [79]. Just as symplectic groupoids are the global objects associated with Poisson manifolds, the twisted symplectic groupoids are the global objects associated with twisted Poisson manifolds [24].

Without non-degeneracy assumptions on ω , one has the following result concerning the infinitesimal version of 3-cocycles [15]: If \mathcal{G} is source-simply connected and $\phi \in \Omega^3(P)$, $d\phi = 0$, then

there is a one-to-one correspondence between 3-cocycles $\omega + \phi$ and bundle maps $\sigma : A \rightarrow T^*P$ satisfying the following two conditions:

$$\langle \sigma(\xi), \rho(\xi') \rangle = -\langle \sigma(\xi'), \rho(\xi) \rangle; \quad (4.7)$$

$$\sigma([\xi, \xi']) = \mathcal{L}_\xi(\sigma(\xi')) - \mathcal{L}_{\xi'}(\sigma(\xi)) + d\langle \sigma(\xi), \rho(\xi') \rangle + i_{\rho(\xi) \wedge \rho(\xi')}(\phi), \quad (4.8)$$

where A is the Lie algebroid of \mathcal{G} , $[\cdot, \cdot]$ is the bracket on $\Gamma(A)$, $\rho : A \rightarrow TP$ is the anchor, and $\xi, \xi' \in \Gamma(A)$. For one direction of this correspondence, given ω , the associated bundle map $\sigma_\omega : A \rightarrow T^*P$ is just $\sigma_\omega(\xi) = i_\xi \omega|_P$.

For a given $\sigma : A \rightarrow T^*P$ satisfying (4.7), (4.8), let us consider the bundle map

$$(\rho, \sigma) : A \rightarrow TP \oplus T^*P. \quad (4.9)$$

A direct computation shows that if the rank of $L_\sigma := \text{Image}(\rho, \sigma)$ equals $\dim(P)$, then $L_\sigma \subset TP \oplus T^*P$ is a ϕ -twisted Dirac structure on P . In this case, it is easy to check that (4.9) yields a (Lie algebroid) isomorphism $A \rightarrow L_\sigma$ if and only if

- 1) $\dim(\mathcal{G}) = 2 \dim(P)$;
- 2) $\ker(\omega_x) \cap \ker(T_x s) \cap \ker(T_x t) = \{0\}$ for all $x \in P$.

A groupoid \mathcal{G} over P satisfying 1) together with a 3-cocycle $\omega + \phi$ so that ω satisfies 2) is called a **ϕ -twisted presymplectic groupoid** [15, 96]. As indicated by the previous discussion, they are precisely the global objects integrating twisted Dirac structures. The 2-form ω is nondegenerate if and only if the associated Dirac structure is Poisson, recovering the known correspondence between (twisted) Poisson structures and (twisted) symplectic groupoids.

The following example describes presymplectic groupoids integrating Cartan-Dirac structures; it is analogous to Example 4.10.

Example 4.14 (*Cartan-Dirac structures and the AMM-groupoid*)

Let G be a Lie group with Lie algebra \mathfrak{g} , equipped with a nondegenerate bi-invariant quadratic form $(\cdot, \cdot)_\mathfrak{g}$. The **AMM groupoid** [8] is the action groupoid $\mathcal{G} = G \ltimes G$ with respect to the conjugation action, together with the 2-form [3]

$$\omega_{(g,x)} = \frac{1}{2} \left((\text{Ad}_x p_g^* \lambda, p_g^* \lambda)_\mathfrak{g} + (p_g^* \lambda, p_x^* (\lambda + \bar{\lambda}))_\mathfrak{g} \right),$$

where p_g and p_x denote the projections onto the first and second components of $G \times G$, and λ and $\bar{\lambda}$ are the left and right Maurer-Cartan forms. The AMM-groupoid is a ϕ^G -twisted presymplectic groupoid integrating L_G [15], the Cartan-Dirac structure on G defined in Example 2.13. If G is simply connected, then $(G \ltimes G, \omega)$ is isomorphic to $\mathcal{G}(L_G)$, the source-simply connected integration of L_G ; in general, one must pull-back ω to $\tilde{G} \ltimes G$, where \tilde{G} is the universal cover of G .

4.3 Morita equivalence for groups and groupoids

Since groupoids play such an important role in the Morita equivalence of Poisson manifolds, we will take some time to discuss Morita equivalence of groupoids in general. We begin with groups.

If we try to define Morita equivalence of groups as equivalence between their (complex linear) representation categories, then we are back to algebra, since representations of a group are the same as modules over its group algebra over \mathbb{C} . (This is straightforward for discrete groups, and more elaborate for topological groups.) Here, we just remark that nonisomorphic groups can have isomorphic group algebras (e.g. two finite abelian groups with the same number of elements), or more generally Morita equivalent group algebras (e.g. two finite groups with the same number of conjugacy classes, hence the same number of isomorphism classes of irreducible representations).

We obtain a more geometric notion of Morita equivalence for groups by considering actions on manifolds rather than on linear spaces. Thus, for Lie groups (including discrete groups) G and H , bimodules are (G, H) -“bispaces”, i.e. manifolds where G acts on the left, H acts on the right, and the actions commute. The “tensor product” of such bimodules is defined by the orbit space

$${}_G X_H * {}_H Y_K := X \times Y / H,$$

where H acts on $X \times Y$ by $(x, y) \mapsto (xh, h^{-1}y)$. The result of this operation may no longer be smooth, even if X and Y are. Under suitable regularity assumptions, to be explained below, the tensor product is a smooth manifold, so we consider the category in which objects are groups and morphisms are isomorphism classes of “regular” bispaces, and we define **Morita equivalence** of groups as isomorphism in this category. Analogously to the case of algebras, we have an associated notion of **Picard group(oid)**.

Exercise

Show that a bispace ${}_G X_H$ is “invertible” if and only if the G and H -actions are free and transitive.

If ${}_G X_H$ is invertible and we fix a point $x_0 \in X$, by the result of the previous exercise, there exists for each $g \in G$ a unique $h \in H$ such that $gx_0 h^{-1} = x_0$. The correspondence $g \mapsto h$ in fact establishes a group isomorphism $G \rightarrow H$. So, for groups, Morita equivalence induces the same equivalence relation as the usual notion of isomorphism. As we will see in Example 4.32 of Section 4.6, the situation for Picard groups resembles somewhat that for algebras, where outer automorphisms play a key role.

For a full discussion of Morita equivalence of Lie groupoids, we refer to the article of Moerdijk and Mrčun [63] in this volume. Here, we will briefly summarize the theory.

An action (from the left) of a Lie groupoid \mathcal{G} over P on a manifold S consists of a map $J : S \rightarrow P$ and a map $\mathcal{G} \times_{(s, J)} S \rightarrow S$ (where s is the source map of \mathcal{G}) satisfying axioms analogous to those of a group action; J is sometimes called the **moment** of the action (see Example 4.16). The action is **principal** with respect to a map $p : S \rightarrow M$ if p is a surjective submersion and if \mathcal{G} acts freely and transitively on each p -fibre; principal \mathcal{G} -bundles are also called **\mathcal{G} -torsors**.

Right actions and torsors are defined in the obvious analogous way. If groupoids \mathcal{G}_1 and \mathcal{G}_2 act on S from the left and right, respectively, and the actions commute, then we call S a $(\mathcal{G}_1, \mathcal{G}_2)$ -**bibundle**. A bibundle is **left principal** when the left \mathcal{G}_1 -action is principal with respect to the moment map for the right action of \mathcal{G}_2 .

If S is a $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundle with moments $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$, and if S' is a $(\mathcal{G}_2, \mathcal{G}_3)$ -bibundle with moments $P_2 \xleftarrow{J'_2} S' \xrightarrow{J'_3} P_3$, then their “tensor product” is the orbit space

$$S * S' := (S \times_{(J_2, J'_2)} S') / \mathcal{G}_2, \quad (4.10)$$

where \mathcal{G}_2 acts on $S \times_{(J_2, J'_2)} S'$ diagonally. The assumption that S and S' are left principal guarantees that $S * S'$ is a smooth manifold and that its natural $(\mathcal{G}_1, \mathcal{G}_3)$ -bibundle structure is left principal.

Two $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundles are **isomorphic** if there is a diffeomorphism between them commuting with the groupoid actions and their moments. The “tensor product” (4.10) is associative up to natural isomorphism, so we may define a category \mathbf{LG} in which the objects are Lie groupoids and morphisms are isomorphism classes of left principal bibundles. Just as in the case of algebras, we call two Lie groupoids **Morita equivalent** if they are isomorphic as objects in \mathbf{LG} , and we define the associated notion of Picard group(oid) just as we do for algebras. We note that a $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundle S is “invertible” in \mathbf{LG} if and only if it is **biprincipal**, i.e., principal with respect to both left and right actions; a biprincipal bibundle is also called a **Morita equivalence** or a **Morita bibundle**.

Example 4.15 (*Transitive Lie groupoids*)

Let \mathcal{G} be a Lie groupoid over P . For a fixed $x \in P$, let \mathcal{G}_x be the isotropy group of \mathcal{G} at x , and let $E_x = s^{-1}(x)$. Then E_x is a $(\mathcal{G}, \mathcal{G}_x)$ -bibundle. It is a Morita bibundle if and only if \mathcal{G} is transitive, i.e., for any $x, y \in P$, there exists $g \in \mathcal{G}$ so that $s(g) = y$ and $t(g) = x$. In fact, a Lie groupoid is transitive if and only if it is Morita equivalent to a Lie group.

4.4 Modules over Poisson manifolds and symplectic groupoid actions

Example 4.1 shows that modules over \mathfrak{g}^* are the same thing as hamiltonian G -manifolds, where G is the connected and simply connected Lie group with Lie algebra \mathfrak{g} . As we discuss in this section, this is a particular case of a much more general correspondence between modules over Poisson manifolds and symplectic groupoid actions.

Let (\mathcal{G}, ω) be a symplectic groupoid over P acting on a symplectic manifold (S, ω_S) with moment J . Let $a : \mathcal{G} \times_{(s, J)} S \rightarrow S$ denote the action. We call the action **symplectic** if it satisfies the property (analogous to the condition on multiplicative forms) that $\text{graph}(a) \subset \mathcal{G}(P) \times S \times \overline{S}$ is lagrangian. Equivalently, a is symplectic if

$$a^* \omega_S = p_S^* \omega_S + p_{\mathcal{G}}^* \omega, \quad (4.11)$$

where $p_S : \mathcal{G} \times_{(s, J)} S \rightarrow S$ and $p_{\mathcal{G}} : \mathcal{G} \times_{(s, J)} S \rightarrow \mathcal{G}$ are the natural projections.

A key observation relating actions of symplectic groupoids to modules over Poisson manifolds is that if $J : S \rightarrow P$ is the moment map for a *symplectic* action of a symplectic groupoid \mathcal{G} over P , then J is automatically a complete Poisson map [60], defining a module over P . On the other hand, a module $J : S \rightarrow P$ over an integrable Poisson manifold P automatically carries a *symplectic* action of the source-simply connected symplectic groupoid $\mathcal{G}(P)$. So there is a one-to-one correspondence between P -modules and symplectic actions of $\mathcal{G}(P)$ [26].

Example 4.16 (*Hamiltonian spaces*)

Let G be a simply-connected Lie group with Lie algebra \mathfrak{g} . Any complete symplectic realization $J : S \rightarrow \mathfrak{g}^*$ induces an action of the symplectic groupoid T^*G on S :

$$\begin{array}{ccc} T^*G & & S \\ \downarrow & \swarrow J & \\ \mathfrak{g}^* & & \end{array}$$

In this case, $T^*G = G \ltimes \mathfrak{g}^*$ is a transformation Lie groupoid, and, as such, its action is equivalent to an ordinary G -action on S for which J is G -equivariant. Moreover, the G -action corresponding to the symplectic T^*G -action induced by $J : S \rightarrow P$ is a hamiltonian G -action for which J is a momentum map. So we recover the result of Example 4.1 on the isomorphism (not only equivalence) between the categories of complete symplectic realizations of \mathfrak{g}^* and hamiltonian G -manifolds. Notice that the momentum map for the group action is the moment map for the groupoid action; it is this example which motivates the term “moment” as applied to groupoid actions.

Remark 4.17 (*Infinitesimal actions*)

The relationship between complete symplectic realizations and symplectic groupoid actions has an infinitesimal counterpart. A symplectic realization (not necessarily complete) $J : S \rightarrow P$ induces a Lie algebra homomorphism

$$\Omega^1(P) \rightarrow \mathcal{X}(S), \quad \alpha \mapsto \tilde{\Pi}_S(J^*\alpha), \quad (4.12)$$

where the bracket on 1-forms is the one of (4.5). This maps defines a Lie algebroid action of the Lie algebroid of P , T^*P , on S . The completeness of J allows this infinitesimal action to be integrated to an action of the source-simply-connected integration $\mathcal{G}(P)$ (see [61]), and this action turns out to be symplectic.

Remark 4.18 (*Symplectic groupoid actions on Poisson manifolds*)

As in Remark 4.2, the correspondence between P -modules and symplectic $\mathcal{G}(P)$ -actions holds in more generality: a Poisson map $Q \rightarrow P$ from *any* Poisson manifold Q induces an infinitesimal T^*P -action on Q , by the same formula as in (4.12). When the Poisson map is complete (and P is integrable), it gives rise to an action of $\mathcal{G}(P)$ on Q , which preserves the symplectic leaves of Q ; its restriction to each leaf is a symplectic action. The action is a Poisson action in the sense that its graph is lagrangian [89] in the appropriate product, see [14] for details.

Remark 4.19 (*Realizations of Dirac structures and presymplectic groupoid actions*)

The correspondence between modules over a Poisson manifold P and symplectic actions of $\mathcal{G}(P)$ extends to one between “modules” over Dirac manifolds and suitable actions of presymplectic groupoids [14, 15].

In order to introduce the notion of “realization” of a Dirac manifold, let us note that, if (P, Π) is a Poisson manifold, then the infinitesimal T^*P -action (4.12) induced by a Poisson map $J : Q \rightarrow P$ can be equivalently expressed in terms of L_Π by the Lie algebra homomorphism

$$\Gamma(L_\Pi) \rightarrow \mathcal{X}(Q), \quad (X, \alpha) \mapsto Y,$$

where Y is uniquely determined by the condition $(Y, J^*\alpha) \in L_{\Pi_Q}$. Since J is a Poisson map, it also follows that $X = TJ(Y)$.

If (P, L) and (Q, L_Q) are Dirac manifolds, and $J : Q \rightarrow P$ is a forward Dirac map, then (2.25) implies that for each $(X, \alpha) \in L$ over the point $J(y) \in P$, there exists $Y \in T_y Q$ such that $(Y, TJ^*(\alpha)) \in (L_Q)_y$ and $X = TJ(Y)$. However, unlike the situation of Poisson maps, Y is *not* uniquely determined by these conditions; this is the case if and only if

$$\ker(TJ) \cap \ker(L_Q) = \{0\}. \quad (4.13)$$

If (4.13) holds at all points of Q , then the induced map $\Gamma(L) \rightarrow \mathcal{X}(Q)$, $(X, \alpha) \mapsto Y$, defines an infinitesimal L -action on Q .

A **Dirac realization** [14] of a ϕ -twisted Dirac manifold (P, L) is a forward Dirac map $J : Q \rightarrow P$, where Q is a $J^*\phi$ -twisted Dirac manifold and (4.13) is satisfied. If Q is a $J^*\phi$ -twisted presymplectic manifold, then J is called a **presymplectic realization**. We call a Dirac realization **complete** if the induced infinitesimal action is complete (in the sense of Lie algebroid actions, see [61]). As in the case of Poisson maps, complete Dirac realizations $J : Q \rightarrow P$ are the same thing as global actions of the presymplectic groupoid $\mathcal{G}(L)$ on Q “compatible” with L_Q in a suitable way [14] (generalizing the conditions in (4.11) and Remark 4.18).

The next example illustrates the discussion in Remark 4.19 and the connection between Dirac geometry and group-valued momentum maps [3, 2].

Example 4.20 (*Modules over Cartan-Dirac structures and quasi-hamiltonian actions*)

As we saw in Example 4.1, symplectic realizations of (resp. Poisson maps into) the Lie-Poisson structure on \mathfrak{g}^* are the same thing as hamiltonian \mathfrak{g} -actions on symplectic (resp. Poisson) manifolds; if the maps are complete, one gets a correspondence with global hamiltonian actions.

Analogously, let us consider a connected, simply-connected Lie group G equipped with L_G , the Cartan-Dirac structure associated with a non-degenerate bi-invariant quadratic form $(\cdot, \cdot)_{\mathfrak{g}}$. Then presymplectic realizations into G are exactly the same as quasi-hamiltonian \mathfrak{g} -manifolds, and complete realizations correspond to global quasi-hamiltonian G -actions (which can be seen as actions of the AMM-groupoid of Example 4.14, analogously to Example 4.16) [15]. More generally, (complete) Dirac realizations of (G, L_G) correspond to (global) hamiltonian quasi-Poisson manifolds [14], in analogy with Remark 4.18.

In these examples, the realization maps are the group-valued momentum maps.

4.5 Morita equivalence of Poisson manifolds and symplectic groupoids

We now have all the ingredients which we need in order to define a geometric notion of Morita equivalence for Poisson manifolds which implies equivalence of their module categories.

A **Morita equivalence** between Poisson manifolds P_1 and P_2 is a (P_1, P_2) -bimodule $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ such that J_1 and J_2 are surjective submersions whose fibres are simply connected and symplectic orthogonals of each other. By Remark 4.6, Morita equivalence only applies to *integrable* Poisson structures. (The nonintegrable case can be handled with the use of symplectic S-groupoids. See Remark 4.40.) The bimodule $P_2 \xleftarrow{J_2} \overline{S} \xrightarrow{J_1} P_1$, where \overline{S} has the opposite symplectic structure, is also a Morita equivalence, and S and \overline{S} satisfy

$$S *_{P_2} \overline{S} \cong \mathcal{G}(P_1), \quad \text{and} \quad \overline{S} *_{P_1} S \cong \mathcal{G}(P_2). \quad (4.14)$$

Since symplectic groupoids are “identity bimodules”, (4.14) is analogous to the invertibility of algebraic bimodules (3.4).

Let us consider the category whose objects are complete symplectic realizations of an integrable Poisson manifold P , and morphisms are symplectic maps between symplectic realizations commuting with the realization maps. This category is analogous to the category of left modules over an algebra, and we call it the **category of modules over P** . If $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ is a Morita equivalence, then the regularity conditions on the maps J_1 and J_2 guarantee that if $S' \rightarrow P_2$ is a left P_2 -module then the tensor product $S *_{P_2} S'$ is smooth and defines a left P_1 -module [94]. So one can define a functor between categories of modules (i.e. complete symplectic realizations) just as one does for algebras, see (3.2) and (3.3), and prove that geometric Morita equivalence implies the equivalence of “representation” categories [52, 94]:

Theorem 4.21 *If P_1 and P_2 are Morita equivalent, then they have equivalent categories of complete symplectic realizations.*

Remark 4.22 *(The “category” of complete symplectic realizations)*

In the spirit of the symplectic “category” of [86], one can also define a larger “category” of complete symplectic realizations of P by considering the morphisms between two P -modules $J : S \rightarrow P$ and $J' : S' \rightarrow P$ to be lagrangian submanifolds in $S' \times_{(J', J)} \overline{S}$, see [93, 94], with composition given by composition of relations; the quotes in “category” are due to the fact that the composition of two such morphisms yields another morphism only under suitable transversality assumptions. Theorem 4.21 still holds in this more general setting [93]. Unlike in the case of algebras, though, the converse of Theorem 4.21 does not hold in general [94], see Remark 4.37. We will discuss ways to remedy this problem in Section 5 by introducing yet another category of representations of P (a “symplectic category”).

Remark 4.23 *(Classical intertwiner spaces)*

As a consequence of (4.14), one can see that Morita equivalence, in addition to establishing an equivalence of module categories, preserves the classical intertwiner spaces.

Remark 4.24 *(More general modules)*

As indicated in Remarks 4.2 and 4.18, from the point of view of hamiltonian actions, it is natural to consider arbitrary complete Poisson maps (not necessarily symplectic realizations) as modules over Poisson manifolds. The “action” of (P_1, P_2) -bimodules on P_2 -modules in Remark 4.4 naturally extends to an action on Poisson maps $Q \rightarrow P_2$; in fact, one can think of this more general tensor product as a leafwise version of the one in Section 4.1, and Theorem 4.21 still holds for these more general “representations”. (This generalization is the analogue, in algebra, of considering homomorphisms of an algebra into direct sums of endomorphism algebras, rather than usual modules.)

The notion of Morita equivalence of Poisson manifolds is closely related to Morita equivalence of symplectic groupoids, which is a refinement of the notion of Morita equivalence for Lie groupoids, taking symplectic structures into account. If \mathcal{G}_1 and \mathcal{G}_2 are symplectic groupoids, then a $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundle is called **symplectic** if both actions are symplectic. The “tensor product” of two symplectic bibundles, as defined in Section 4.3, is canonically symplectic, so we may define a category \mathbf{SG} in which the objects are symplectic groupoids and morphisms are isomorphism classes of left principal symplectic bibundles. (An isomorphism between symplectic bibundles is required to preserve the symplectic forms.) We call two symplectic groupoids \mathcal{G}_1 and \mathcal{G}_2 **Morita equivalent** [94] if they are isomorphic in \mathbf{SG} , i.e. if there exists a biprincipal symplectic $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundle (see [52]). A **Morita equivalence** between symplectic groupoids is a symplectic bibundle which is biprincipal.

If P_1 and P_2 are Poisson manifolds, and if $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ is a (P_1, P_2) -bimodule, then we obtain a left symplectic action of the groupoid $\mathcal{G}(P_1)$ and right symplectic action of $\mathcal{G}(P_2)$,

$$\begin{array}{ccccc} \mathcal{G}(P_1) & & S & & \mathcal{G}(P_2) \\ \Downarrow & \swarrow J_1 & & \searrow J_2 & \Downarrow \\ P_1 & & & & P_2 \end{array}$$

The property that $\{J_1^* C^\infty(P_1), J_2^* C^\infty(P_2)\} = 0$ implies that the actions of $\mathcal{G}(P_1)$ and $\mathcal{G}(P_2)$ commute, so that S is a symplectic $(\mathcal{G}(P_1), \mathcal{G}(P_2))$ -bibundle. We say that a symplectic bimodule

$P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ is **regular** if the associated symplectic $(\mathcal{G}(P_1), \mathcal{G}(P_2))$ -bibundle is left principal. The tensor product of symplectic bimodules defined in Section 4.1 coincides with their tensor product as symplectic bibundles. As a result, the tensor product of regular symplectic bimodules is smooth and regular.

Remark 4.25 (*Regular bimodules*)

Regular bimodules can be described with no reference to the symplectic groupoid actions: $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ is regular if and only if J_1 and J_2 are complete Poisson maps, J_1 is a submersion, J_2 is a surjective submersion with simply-connected fibres, and the J_1 - and J_2 -fibres are symplectic orthogonal of one another.

Exercise

Prove the equivalent formulation of regular bimodules in Remark 4.25. (Hint: this is a slight extension of [93, Thm. 3.2])

We define the category **Poiss** in which the objects are integrable Poisson manifolds and morphisms are isomorphism classes of regular symplectic bimodules.

If $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ is a Morita equivalence of Poisson manifolds, then the regularity assumptions on the maps J_1 and J_2 insure that S is biprincipal for the induced actions of $\mathcal{G}(P_1)$ and $\mathcal{G}(P_2)$, so that S is also a Morita equivalence for the symplectic groupoids $\mathcal{G}(P_1)$ and $\mathcal{G}(P_2)$. On the other hand, if \mathcal{G}_1 and \mathcal{G}_2 are source-simply-connected symplectic groupoids over P_1 and P_2 , respectively, then a $(\mathcal{G}_1, \mathcal{G}_2)$ -Morita equivalence is a (P_1, P_2) -Morita equivalence. So two integrable Poisson manifolds P_1 and P_2 are Morita equivalent if and only if their source-simply-connected integrations, $\mathcal{G}(P_1)$ and $\mathcal{G}(P_2)$, are Morita equivalent as symplectic groupoids.

Remark 4.26 (*Lie functor*)

It follows from the discussion above that there exists a natural equivalence between the category of *source-simply-connected* symplectic groupoids with morphisms being Morita equivalences (resp. left principal symplectic bibundles), and the category of integrable Poisson manifolds with morphisms being Morita equivalences (resp. regular bimodules). These equivalences are similar to the one between the categories of Lie algebras and simply-connected Lie groups, with their usual morphisms.

Example 4.27 (*Symplectic manifolds*)

Let P be a connected symplectic manifold. The universal cover of P with base point x , denoted \tilde{P} , is a Morita equivalence between the symplectic groupoid $\mathcal{G}(P)$, which in this case is the fundamental groupoid of P , and $\pi_1(P, x)$:

$$\begin{array}{ccc} \mathcal{G}(P) & \tilde{P} & \pi_1(P, x) \\ \Downarrow & \swarrow \quad \searrow & \Downarrow \\ P & & \{x\} \end{array} \tag{4.15}$$

Note that $\pi_1(P, x)$ is a symplectic groupoid for the zero-dimensional Poisson manifold $\{x\}$, though generally not the source-simply-connected one.

In analogy with Example 4.16 on hamiltonian actions: there is an equivalence of categories between complete symplectic realizations of P and symplectic actions of $\pi_1(P, x)$. This suggests

the slogan that “a (connected) symplectic manifold P with fundamental group $\pi_1(P)$ is the dual of the Lie algebra of $\pi_1(P)$ ”.

It follows from the Morita equivalence (4.15) and the discussion about Morita equivalence of groups in Section 3.1 that connected symplectic manifolds P_1 and P_2 are Morita equivalent if and only if $\pi_1(P_1) \cong \pi_1(P_2)$.

Example 4.28 (*Symplectic fibrations*)

It follows from the previous example that every simply-connected symplectic manifold is Morita equivalent to a point. Similarly, if (Q, Π) is a Poisson manifold with $\Pi = 0$, then Q is Morita equivalent to any product $Q \times S$ where S is a simply-connected symplectic manifold. In fact, $Q \times S \xleftarrow{\text{pr}_1} T^*Q \times S \xrightarrow{\text{pr}_2} Q$ is a Morita bimodule, where pr_1 and pr_2 are the natural projections.

More generally, let us assume that P is a Poisson manifold whose symplectic foliation is a fibration $P \rightarrow Q$ with simply-connected fibres. In general, there are obstructions to P being Morita equivalent to Q [93]: P is Morita equivalent to Q if and only if there exists a closed 2-form on P which restricts to the symplectic form on each fibre. We will have more to say about “fibrating” Poisson manifolds and their Morita invariants in Section 4.7.

Example 4.29 (*Lie-Poisson structures*)

Let us consider \mathfrak{g}_1^* and \mathfrak{g}_2^* , the duals of the Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , equipped with their Lie-Poisson structures. Then \mathfrak{g}_1^* and \mathfrak{g}_2^* are Morita equivalent if and only if they are isomorphic. Indeed, suppose that $\mathfrak{g}_1^* \xleftarrow{J_1} S \xrightarrow{J_2} \mathfrak{g}_2^*$ is a Morita bimodule, and let $X = J_2^{-1}(0)$. A dimension count shows that there exists $\mu \in \mathfrak{g}_1^*$ such that $X = J_1^{-1}(\mu)$. Since S is a biprincipal bibundle for the symplectic groupoids $\mathcal{G}(\mathfrak{g}_i^*) = T^*G_i$, $i = 1, 2$, it follows that X is a (G_1, G_2) -Morita bibundle. Therefore G_1 and G_2 are isomorphic, and so are \mathfrak{g}_1^* and \mathfrak{g}_2^* .

This example also follows from the Morita invariants discussed in Section 4.7.

Example 4.30 (*Topologically stable structures on surfaces*)

Let Σ be a compact, connected, oriented surface equipped with a Poisson structure Π which has at most linear degeneracies and whose zero set consists of n smooth, disjoint, closed curves, for $n \geq 0$. These are called **topologically stable structures** (TSS) [70].

Any two modular vector fields for Π [91] coincide at points where Π vanishes, so the curves in the zero set carry a natural orientation. We denote the zero set of Π , regarded as an oriented 1-manifold, by $Z(\Sigma, \Pi)$. Two TSS (Σ, Π) and (Σ, Π') are **topologically equivalent** if there is an orientation-preserving diffeomorphism $\psi : \Sigma \rightarrow \Sigma'$ such that $\psi(Z(\Sigma, \Pi)) = Z(\Sigma', \Pi')$. We denote the equivalence class of $Z(\Sigma, \Pi)$ by $[Z(\Sigma, \Pi)]$. This class can be represented by an oriented labeled graph $\mathfrak{G}(\Sigma, \Pi)$: each vertex corresponds to a 2-dimensional leaf of the structure, two vertices being connected by an edge for each boundary zero curve they share; each edge is oriented to point toward the vertex for which Π is positive with respect to the orientation of Σ . We then label each vertex by the genus of the corresponding leaf.

It turns out that the topology of the zero set plus the modular periods (periods of a modular vector field around the zero curves) completely determine the Morita equivalence class of TSS [16, 19]. In fact, let us define a more elaborate graph $\mathfrak{G}_T(\Sigma, \Pi)$, obtained from $\mathfrak{G}(\Sigma, \Pi)$ by labeling each of its edges by the modular period around the corresponding zero curve. Then two TSS (Σ, Π) and (Σ', Π') are Morita equivalent if and only if there is an isomorphism of labeled graphs $\mathfrak{G}_T(\Sigma, \Pi) \cong \mathfrak{G}_T(\Sigma', \Pi')$. (It follows from the results in [29] that TSS are always integrable.)

The classification of TSS up to Morita equivalence was preceded by (and builds on) their classification up to orientation-preserving Poisson diffeomorphisms by Radko [70], who shows that the topological class of the zero set and the modular periods, together with a certain volume invariant (generalizing the Liouville volume when the TSS is symplectic), form a complete set of invariants.

Remark 4.31 (*Morita equivalence of presymplectic groupoids and “momentum map theories”*)

As we noted in Examples 4.16 and 4.20, hamiltonian spaces can be seen as modules over Lie-Poisson structures on duals of Lie algebras, whereas quasi-hamiltonian (or hamiltonian quasi-Poisson) manifolds are modules over Cartan-Dirac structures on Lie groups. Thus, the category of modules over an arbitrary (integrable) Poisson or Dirac manifold can be regarded as the category of “hamiltonian spaces” for some generalized “momentum map theory”. Since Morita equivalence establishes an equivalence of categories of modules, it provides a precise notion of equivalence for “momentum map theories” and automatically implies the existence of other invariants (such as classical intertwiner spaces—see Remark 4.23).

An extended notion of Morita equivalence for ϕ -twisted presymplectic groupoids (or, infinitesimally, ϕ -twisted Dirac structures) was developed by Xu in [96]. In Xu’s work, it is shown that various known correspondences of “momentum map theories” can be described by appropriate Morita equivalences. Examples include the equivalence between ordinary momentum maps and momentum maps for actions of Poisson-Lie groups (taking values in the dual group) [1, 41] and the one between quasi-hamiltonian spaces for groups and ordinary hamiltonian spaces for their loop groups [3]. An interesting feature of Morita equivalence for presymplectic groupoids is that the bimodules are not simply a pair of module structures which commute.

Besides relating “momentum map theories”, Morita equivalence of groupoids plays a central role in certain approaches to geometric quantization of these generalized hamiltonian spaces, where the usual line bundles are replaced by gerbes [8, 53].

4.6 Picard groups

Just as for algebras, there are Picard groupoids associated with the categories **Poiss** and **SG**. In particular, the isomorphism classes of Morita self-equivalences of a Poisson manifold P (resp. symplectic groupoid \mathcal{G}) form a group $\text{Pic}(P)$ (resp. $\text{Pic}(\mathcal{G})$), called the **Picard group**. It follows from the discussion in the previous section that $\text{Pic}(P) = \text{Pic}(\mathcal{G}(P))$.

We now discuss some examples of “geometric” Picard groups; see [19] for details.

Example 4.32 (*Picard groups of groups*)

As we saw in Section 4.3, geometric Morita equivalences between groups are closely related to group isomorphisms. A closer analysis shows that the Picard group of a group G is naturally isomorphic to its group $\text{OutAut}(G) := \text{Aut}(G)/\text{InnAut}(G)$ of outer automorphisms.

It follows from Example 4.15 and the invariance of Picard groups under Morita equivalence that, if \mathcal{G} is a transitive groupoid over P , then $\text{Pic}(\mathcal{G}) \cong \text{OutAut}(\mathcal{G}_x)$, where \mathcal{G}_x is the isotropy group at a point $x \in P$. This isomorphism is natural, so the outer automorphism groups attached to different points are all naturally isomorphic to one another.

Example 4.33 (*Picard groups of symplectic manifolds*)

Since, according to Example 4.27, the fundamental groupoid of a connected symplectic manifold P is Morita equivalent to any of its fundamental groups $\pi_1(P, x)$, it follows from Example

4.32 that, for such a manifold, $\text{Pic}(P)$ is naturally isomorphic to $\text{OutAut}(\pi_1(P, x))$ for any x in P .

The Picard group of a Poisson manifold or symplectic manifold is also related to a group of outer automorphisms of the manifold itself. For a Poisson manifold P , let $\text{Aut}(P)$ denote its group of Poisson diffeomorphisms. There is a natural map

$$j : \text{Aut}(P) \rightarrow \text{Pic}(P), \quad (4.16)$$

analogous to (3.8), which assigns to each $\psi \in \text{Aut}(P)$ the isomorphism class of the bimodule $P \xleftarrow{t} \mathcal{G}(P) \xrightarrow{\psi^{-1} \circ s} P$. Any lagrangian bisection of $\mathcal{G}(P)$ (which is the analogue of a group element) naturally induces a Poisson diffeomorphism of P that we call an **inner automorphism**. It turns out that $\ker(j) = \text{InnAut}(P)$, the group of inner automorphisms of P , just as in the algebraic setting discussed in Section 3.1.

The situation for symplectic groupoids is completely analogous [19].

Exercise

Let P be the $2n$ dimensional torus $\mathbb{R}^{2n}/(2\pi\mathbb{Z})^{2n}$ with a symplectic structure of the form $\frac{1}{2}\omega_{ij}d\theta^i \wedge d\theta^j$, where ω is a nondegenerate antisymmetric matrix of real constants. Show that the Picard group of P is independent of the choice of ω , while the subgroup of $\text{Pic}(P)$ arising from outer automorphisms of (P, ω) does depend on ω .

Exercise

Compare $\text{OutAut}(P)$ with $\text{Pic}(P)$ when P is the disjoint union of several 2-dimensional spheres, possible with different symplectic areas. Hint: use the theorems of Smale [80] and Moser [65] to show that every symplectomorphism of S^2 is inner.

There are geometric versions of the maps (3.7) and (3.9). Let $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ be a Morita equivalence. If $\mathcal{O} \subseteq P_2$ is a symplectic leaf, then $J_1(J_2^{-1}(\mathcal{O}))$ is a symplectic leaf of P_1 , and this is a bijective correspondence between symplectic leaves. So, for a Poisson manifold P , we have a map

$$\text{Pic}(P) \rightarrow \text{Aut}(\text{Leaf}(P)), \quad (4.17)$$

where $\text{Leaf}(P)$ is the leaf space of P , analogous to the map (3.9). We define the **static Picard group** $\text{SPic}(P)$ of P as the kernel of (4.17), i.e., the self-Morita equivalences inducing the identity map on the leaf space. Note that functions on the leaf space constitute the center of the Poisson algebra of functions on P , hence the terminology analogous to that for algebras.

Example 4.34 (*Zero Poisson structures*)

As we saw in Example 4.9, in this case $\mathcal{G}(P) = T^*P$, and $\text{Pic}(P) = \text{Pic}(T^*P)$. Since $\text{Leaf}(P) = P$, (4.17) implies that each self-Morita bimodule S induces a diffeomorphism ψ of P . So composing S with ψ^{-1} defines an element of the static Picard group $\text{SPic}(P)$. A direct computation shows that the map (4.17) is split by the map $\text{Aut}(P) \rightarrow \text{Pic}(P)$ (4.16), hence

$$\text{Pic}(P) = \text{Diff}(P) \ltimes \text{SPic}(P),$$

in complete analogy with (3.10). Bimodules in $\text{SPic}(P)$ are of the form

$$\begin{array}{c} S \\ \downarrow \scriptstyle P \quad \downarrow \scriptstyle P \\ P \end{array} \quad (4.18)$$

so each fibre $p^{-1}(x)$ is lagrangian and simply-connected; moreover, the fact that p is a *complete* Poisson map implies that the p -fibres are complete with respect to their natural affine structure.

Since $P \leftarrow S \rightarrow P$ is a Morita bimodule, the p -fibres are isomorphic to the fibres of the symplectic groupoid target map $T^*P \rightarrow P$, so they are contractible. As a result, there exists a cross section $P \rightarrow S$, which implies that there is a diffeomorphism $S \cong T^*P$ preserving the fibres [19, Sec. 3]. Hence, in order to characterize a bimodule (4.18), the only remaining freedom is on the choice of symplectic structure on T^*P . It turns out that the most general possible symplectic structure on T^*P with respect to which the fibres of $T^*P \rightarrow P$ are lagrangian and complete is of the form:

$$\omega + p^*B,$$

where ω is the canonical symplectic form on T^*P and B is a closed 2-form on P (a “magnetic” term). One can show that two such bimodules are isomorphic if and only if B is exact. Hence

$$\mathrm{SPic}(P) \cong H^2(P, \mathbb{R}), \quad (4.19)$$

and

$$\mathrm{Pic}(P) \cong \mathrm{Diff}(P) \ltimes H^2(P, \mathbb{R}), \quad (4.20)$$

where the semi-direct product is with respect the natural action of $\mathrm{Diff}(P)$ on $H^2(P, \mathbb{R})$ by pull back. The reader can find the details in [19, Sec. 6.2].

Remark 4.35 (*An intriguing resemblance*)

Recall from Example 3.5 that the Picard group of the algebra $C^\infty(P)$ (which can be seen as a trivial quantization of (P, Π) , if $\Pi = 0$) is $\mathrm{Diff}(P) \ltimes H^2(P, \mathbb{Z})$. Is there a theorem relating classical and quantum Picard groups which would explain the similarity between this fact and (4.20)?

4.7 Fibrating Poisson manifolds and Morita invariants

In this section, we will discuss “rigidity” aspects of geometric Morita equivalence. As we saw in Theorem 4.21, Morita equivalence preserves categories of “geometric representations”. We point out a few other invariants, some of which have already appeared in previous sections.

1. As shown in Example 4.27, the Morita equivalence class of a symplectic manifold is completely determined by the isomorphism class of its fundamental group;
2. As remarked in Section 4.6, Morita equivalence induces a one-to-one correspondence of symplectic leaves, which is a diffeomorphism whenever the leaf spaces are smooth; moreover, corresponding symplectic leaves are themselves Morita equivalent [16, 30] and have isomorphic transverse Poisson structures [87];
3. Morita equivalence preserves first Poisson cohomology groups [28, 40], and modular classes [28, 91];
4. The monodromy groups and isotropy Lie algebras are Morita invariant [30].

As remarked in [30], all the invariants listed above turn out to be preserved by a notion of equivalence which is much weaker than Morita equivalence, called **weak Morita equivalence**, which does not require the integrability of Poisson manifolds. We do not know any Morita invariant which is not a weak Morita invariant.

By 1. above, the only Morita invariant of a connected symplectic manifold is its fundamental group. For a disjoint union of symplectic components, it is the unordered list of fundamental groups which counts; in particular, if all the components are simply connected, the number of components is a complete invariant. In this section, we will see that the Morita invariant structure is much richer for a Poisson manifold which is a smooth family of (diffeomorphic) symplectic manifolds.

We will say that a Poisson manifold P is **fibrating** if its symplectic leaves are the fibres of a smooth locally trivial fibration from P to $\text{Leaf}(P)$. Here, locally triviality is meant in the differentiable rather than symplectic sense; in fact, it is the variation in symplectic structure from fibre to fibre which will concern us.

When P is fibrating, the fibrewise homology groups $H_2(\text{Fib}, \mathbb{Z})$ form a locally trivial bundle of abelian groups over $\text{Leaf}(P)$. Pairing with the fibrewise symplectic structure gives a map $H_2(\text{Fib}, \mathbb{Z}) \rightarrow \mathbb{R}$, which encodes the variation of the symplectic cohomology class from fibre to fibre. The derivative of this map with respect to the base point in $\text{Leaf}(P)$ gives rise to a map $\nu : H_2(\text{Fib}, \mathbb{Z}) \rightarrow T^*\text{Leaf}(P)$.

The map ν vanishes on torsion elements of $H_2(\text{Fib}, \mathbb{Z})$, so its image is a family of embedded abelian groups in the fibres of $T^*\text{Leaf}(P)$, called the **variation lattice** of P . Dazord [33] proves that, if P is integrable and has simply connected fibres, the variation lattice must be topologically closed with constant rank, having local bases of closed 1-forms. Failure of the variation lattice to have these properties provides an obstruction to integrability which was extended to general Poisson manifolds in [30].

A nice application of the variation lattice is to the study of the Picard groups of the duals of Lie algebras of compact groups [19], in which the lattice imparts a flat affine structure to the regular part of the symplectic leaf space.

Example 4.36 (*Nonintegrable Poisson structures revisited*)

Let us again consider $P = \mathbb{R} \times S^2$ from Example 4.12, with Poisson structure $(1/f(t))\Pi_{S^2} \times 0$, $f(t) > 0$. The area of the symplectic leaf over $t \in \mathbb{R}$ is $4\pi f(t)$. The variation lattice is spanned by $4\pi f'(t)dt$, so it has constant rank if and only if $f'(t) \equiv 0$ or $f'(t)$ is not zero for all t .

If P_1 and P_2 are Morita equivalent fibrating Poisson manifolds with simply connected leaves, then the induced diffeomorphism $\text{Leaf}(P_1) \rightarrow \text{Leaf}(P_2)$ preserves the variation lattice; this can be seen as a special case of 4. above. So, although Morita equivalence does not determine the fibrewise symplectic structures, it is sensitive to how symplectic structures vary from fibre to fibre. This sensitivity leads to the following example [93], of Poisson manifolds which are representation equivalent but not Morita equivalent (see Remark 4.22).

Example 4.37 (*Representation equivalence vs. Morita equivalence*)

Consider $(0, 1) \times S^2$ with Poisson structures Π_1 and Π_2 determined by the fibrewise symplectic structures $(1/t)\Pi_{S^2}$ and $(1/2t)\Pi_{S^2}$, respectively. Their variation lattices are spanned by $4\pi dt$ and $8\pi dt$, respectively. Since there is no diffeomorphism of $(0, 1)$ taking dt to $2dt$, these structures cannot be Morita equivalent. Note however that these structures are representation equivalent: representations of Π_1 and Π_2 can be interchanged by dividing or multiplying the symplectic form on the realizations by 2.

Remark 4.38 (*A complete invariant?*)

Xu [93] shows that the leaf space with its variation lattice completely determines the Morita equivalence class of a fibrating Poisson manifold for which the symplectic leaves are simply

connected and form a differentiably *globally* trivial fibration. It does not seem to be known whether this result persists without the global triviality assumption. The attempt to attack this problem by “gluing” together applications of the known case to local trivializations seems to lead to the problem of computing the static Picard group of a fibrating Poisson manifold.

To extend the discussion above to the case where the leaves are not simply connected, it seems that the variation lattice should be replaced by its “spherical” part, obtained by replacing $H_2(\text{Fib}, \mathbb{Z})$ by the subgroup consisting of the spherical classes, i.e. the image of the Hurewicz homomorphism from the bundle $\pi_2(\text{Fib})$ of homotopy groups. This spherical variation lattice is very closely related to the monodromy groups in [30]. Details in this case should be interesting to work out, particularly when the symplectic leaf fibration is not globally trivial.

Remark 4.39 (*Noncompact fibres*)

If the leaves of a fibrating Poisson manifold are compact, Moser’s theorem [65] implies that the variation lattice actually measures how the isomorphism class of the symplectic structure varies from leaf to leaf. If the leaves are noncompact, e.g. if they are discs in \mathbb{R}^2 , then their area can vary without this being detected by any Morita invariant. Is there another notion of Morita invariance which would detect the variation from fibre to fibre of symplectic volume or other invariants, such as capacities?

Remark 4.40 (*Morita equivalence for nonintegrable Poisson manifolds*).

For a fibrating Poisson manifold which is nonintegrable, the variation lattice still exists, so one might hope that it is still a Morita invariant when the leaves are simply connected. But there is no Morita equivalence between such a manifold and itself, much less another one. To remedy this problem, we should extend the notion of Morita equivalence to admit as bimodules smooth stacks which are not manifolds, as we did for self-equivalences in Section 4.2. If we do this, then the variation lattice is indeed Morita invariant. In particular, this shows that integrability is an invariant property under this broadened notion of Morita invariance. Moreover, it turns out that any “S”-Morita equivalence between integrable Poisson manifolds is given by a manifold, so that the integrable part of the Picard groupoid remains unchanged. It would be interesting to see how S-Morita equivalence is related to weak Morita equivalence.

4.8 Gauge equivalence of Poisson structures

Let P be a manifold, and let $\phi \in \Omega^3(P)$ be closed. There is a natural way in which *closed* 2-forms on P act on ϕ -twisted Dirac structures: if $B \in \Omega^2(P)$ is closed and L is a ϕ -twisted Dirac structure on P , then we set

$$\tau_B(L) := \{(X, \alpha + \tilde{B}(X)) \mid (X, \alpha) \in L\},$$

which is again a ϕ -twisted Dirac structure. We call this operation on Dirac structures a **gauge transformation** associated with a 2-form [79]. (More generally, for an arbitrary B , $\tau_B(L)$ is a $(\phi - dB)$ -twisted Dirac structure.) Geometrically, a gauge transformation changes a Dirac structure L by adding the pull-back of a closed 2-form to its leafwise presymplectic form.

Remark 4.41 (*Gauge transformations and B-fields*)

In a complete similar way, complex closed 2-forms act on complex Dirac structures. If $B \in \Omega^2(P)$ is a *real* 2-form, and L is a generalized complex structure on P (see Remark 2.11), then one can show that $\tau_B(L)$ is again a generalized complex structure, and this operation is called a **B-field transform** [42, 45].

If Π is a Poisson structure on P , then changing it by a gauge transformation will generally result in a Dirac structure which is no longer Poisson. In fact, if $B \in \Omega^2(P)$ is closed, then $\tau_B(L_\Pi)$ is a Poisson structure if and only if the bundle map

$$\text{Id} + \tilde{B}\tilde{\Pi} : T^*P \rightarrow T^*P \quad (4.21)$$

is invertible. In this case, the resulting Poisson structure is the one associated with the bundle map

$$\tilde{\Pi}(\text{Id} + \tilde{B}\tilde{\Pi})^{-1} : T^*P \rightarrow TP,$$

and we denote it by $\tau_B(\Pi)$.

Let (P, Π) be a fibrating Poisson manifold, as in Section 4.7. Since a gauge transformation adds the pull-back of a closed 2-form on P to the symplectic form on each fibre, the cohomology classes of fibrewise symplectic forms may change in this operation; however, the way they vary from fibre to fibre does not. This suggests that gauge transformations preserve the Morita equivalence class of (P, Π) . In fact, this holds in complete generality [16]:

Theorem 4.42 *Gauge equivalence of integrable Poisson structures implies Morita equivalence.*

Since gauge transformations do not change the foliation of a Poisson structure, there is no hope that the converse of Theorem 4.42 holds, since even Poisson diffeomorphic structures may have different foliations. We call two Poisson manifolds (P_1, Π_1) and (P_2, Π_2) **gauge equivalent up to Poisson diffeomorphism** if there exists a Poisson diffeomorphism $\psi : (P_1, \Pi_1) \rightarrow (P_2, \tau_B(\Pi_2))$ for some closed 2-form $B \in \Omega^2(P_2)$. It clearly follows from Theorem 4.42 that if two integrable Poisson manifolds are gauge equivalent up to a Poisson diffeomorphism, then they are Morita equivalent. The following properties are clear:

1. Two symplectic manifolds are gauge equivalent up to Poisson diffeomorphism if and only if they are symplectomorphic;
2. If two Poisson manifolds are gauge equivalent up to Poisson diffeomorphism, then they have isomorphic foliations (though generally different leafwise symplectic structures);
3. The Lie algebroids associated with gauge equivalent Dirac structures are isomorphic [79]; as a result, two Poisson manifolds which are gauge equivalent up to Poisson diffeomorphism have isomorphic Poisson cohomology groups.

A direct comparison between the properties above and the Morita invariants listed in Section 4.7 suggests that Morita equivalence should be a weaker notion of equivalence. Indeed, two nonisomorphic symplectic manifolds with the same fundamental group are Morita equivalent, but not gauge equivalent up to Poisson diffeomorphism. In [16, Ex. 5.2], one can also find examples of Morita equivalent Poisson structures on the *same* manifold which are not gauge equivalent up to Poisson diffeomorphism by finding nonequivalent symplectic fibrations with diffeomorphic total space and base (and using Example 4.28). Nevertheless, there are interesting classes of Poisson structures for which both notions of equivalence coincide, such as the topologically stable structures of Example 4.30 [16, 19].

Remark 4.43 (*Gauge transformations and Morita equivalence of quantum algebras*)

As mentioned in Remark 3.14, gauge transformations associated with *integral* 2-forms define an action of $H^2(P, \mathbb{Z})$ on formal Poisson structures on P which can be “quantized” (via Kontsevich’s quantization [48]) to Morita equivalent deformation quantization algebras.

On the other hand, gauge transformations of translation-invariant Poisson structures on tori are particular cases of the linear fractional transformations (3.19), which quantize, according to Theorem 3.15, to strongly Morita equivalent quantum tori. As we already mentioned in Remark 4.35, it would be very interesting to have a unified picture relating Morita equivalence of quantum algebras to geometric Morita equivalence.

5 Geometric representation equivalence

In Section 4, we considered the category of P -modules (i.e. complete symplectic realizations) over a Poisson manifold P , the geometric analogue of the category of left modules over an algebra. We observed in Remark 4.37 that, unlike the category of representations of an algebra, this category does not determine the Morita equivalence class of P . In this section, we will discuss refinements of the notion of category of representations of a Poisson manifold in order to remedy this defect.

5.1 Symplectic torsors

The first refinement we discuss is motivated by the theory of differentiable stacks [7, 59, 69].

Given a Lie groupoid \mathcal{G} , let $B\mathcal{G}$ denote the category of (left) \mathcal{G} -torsors. If two Lie groupoids \mathcal{G}_1 and \mathcal{G}_2 are Morita equivalent, then the natural functor $B\mathcal{G}_1 \rightarrow B\mathcal{G}_2$ induced by any Morita bibundle establishes an equivalence of these categories.

However, to recover the Morita equivalence class of \mathcal{G} from $B\mathcal{G}$, one needs to consider another piece of information: the natural “projection” functor $B\mathcal{G} \rightarrow \mathcal{C}$, where \mathcal{C} denotes the category of smooth manifolds, which assigns to a \mathcal{G} -torsor $S \rightarrow M$ the manifold $M = S/\mathcal{G}$. The category $B\mathcal{G}$ *together* with this projection functor is an example of a differentiable stack. Taking this extra structure into account, one defines $B\mathcal{G}_1$ and $B\mathcal{G}_2$ to be **isomorphic** if there is an equivalence of categories $B\mathcal{G}_1 \rightarrow B\mathcal{G}_2$ commuting with the respective “projections” into \mathcal{C} .

It is clear that a functor induced by a Morita bibundle establishes an isomorphism of stacks of torsors. It turns out that the converse is also true: if $B\mathcal{G}_1$ and $B\mathcal{G}_2$ are isomorphic in this refined sense, then the Lie groupoids \mathcal{G}_1 and \mathcal{G}_2 are Morita equivalent. As we will see, much of this discussion can be adapted to the context of Poisson manifolds and symplectic groupoids.

Let P be an integrable Poisson manifold. A **symplectic P -torsor** is a complete symplectic realization $J : S \rightarrow P$ with the additional property that the induced left action of the symplectic groupoid $\mathcal{G}(P)$ on S is principal. Note that, in this case, the manifold $M = S/\mathcal{G}(P)$ has a natural Poisson structure. (As with the regular bimodules in Remark 4.25, we can also describe symplectic torsors without reference to groupoids. $J : S \rightarrow P$ should be a surjective submersion, and the symplectic orthogonal leaves to the J -fibres should be simply-connected and form a simple foliation.)

Instead of considering the category of all complete symplectic realizations over a Poisson manifold P , let us consider the category BP of symplectic P -torsors, as we did for Lie groupoids. If we restrict the morphisms in BP to symplectomorphisms, then there is a well-defined “projection” functor $BP \rightarrow \mathcal{C}_{Pois}$, where \mathcal{C}_{Pois} denotes the category of Poisson manifolds, with *ordinary* (invertible) Poisson maps as morphisms. As in the case of Lie groupoids, we refine the notion of isomorphism of categories to include the “projection” functors: BP_1 and BP_2 are **isomorphic** if there is an equivalence of categories $BP_1 \rightarrow BP_2$ commuting with the projections $BP_i \rightarrow \mathcal{C}_{Pois}$, $i = 1, 2$. In this setting, it is also clear that a Morita equivalence of P_1 and P_2 induces an isomorphism between BP_1 and BP_2 . The following is a natural question.

If BP_1 and BP_2 are isomorphic, must P_1 and P_2 be Morita equivalent Poisson manifolds?

In Remark 4.37, we saw that the Poisson manifolds $P_1 = ((0, 1) \times S^2, (1/t)\Pi_{S^2})$ and $P_2 = ((0, 1) \times S^2, (1/2t)\Pi_{S^2})$ are not Morita equivalent, but there is an equivalence of categories $BP_1 \rightarrow BP_2$. However, this equivalence does *not* commute with the “projection” functors, so it is not an isomorphism in the refined sense. Thus there is some hope that the answer to the question above is “yes,” though we do not yet have a complete proof.

5.2 Symplectic categories

The next approach to find a “category of representations” that determines the Morita equivalence class of a Poisson manifold is based on the notion of “symplectic category”. One can think of it as the classical limit of the usual notion of abelian category, in the sense that the vector spaces (or modules) of morphisms in the theory of abelian categories are replaced by symplectic manifolds. Notice that we are referring to *a* “symplectic category”, rather than *the* symplectic “category” of [86]. From now on, we will drop the quotation marks when referring to the new notion.

In a **symplectic category**, one has a class of objects, and, for any two objects A and B , a symplectic manifold, denoted by $\text{Hom}(A, B)$, which plays the role of the space of morphisms from B to A . Given three objects A , B and C , the “composition operation” $\text{Hom}(A, C) \leftarrow \text{Hom}(A, B) \times \text{Hom}(B, C)$ is a lagrangian submanifold

$$L_{ABC} \subset \text{Hom}(A, C) \times \overline{\text{Hom}(A, B)} \times \overline{\text{Hom}(B, C)}.$$

This may *not* be the graph of a map, but just a canonical relation, so we will refer to it as the **composition relation**. So, unlike in ordinary categories, $\text{Hom}(A, B)$ should not be thought of as a set of points. Instead, certain lagrangian submanifolds of $\text{Hom}(A, B)$ will play the role of “invertible elements”, so that we can talk about “isomorphic” objects. In other words, the guiding principle is to think of *a* symplectic category as a category in *the* symplectic “category.”

A **functor** between symplectic categories should consist of a map F between objects together with symplectic maps $\text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$, so that the induced map from

$$\text{Hom}(A, C) \times \text{Hom}(A, B) \times \text{Hom}(B, C)$$

to

$$\text{Hom}(F(A), F(C)) \times \text{Hom}(F(A), F(B)) \times \text{Hom}(F(B), F(C))$$

maps the composition relation L_{ABC} to $L_{F(A)F(B)F(C)}$. It is also natural to require that if $\text{Hom}(A, B)$ contains “invertible elements”, then so does $\text{Hom}(F(A), F(B))$.

If \mathcal{S} and \mathcal{S}' are symplectic categories, then a functor $\mathcal{S} \rightarrow \mathcal{S}'$ is an **equivalence of symplectic categories** if for any object A' in \mathcal{S}' , there exists an object A such that $F(A)$ and A' are “isomorphic” (in the sense that $\text{Hom}(F(A), A')$ contains an “invertible element”), and the maps $\text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$ are symplectomorphisms.

We have not answered some questions about symplectic categories which arise naturally. Is $\text{Hom}(A, A)$ always a symplectic groupoid? If not, what are sufficient conditions? Is there always a “base” functor from a given symplectic category to the category of Poisson manifolds and Morita morphisms? Nevertheless, we can still discuss interesting examples, such as the one which follows.

5.3 Symplectic categories of representations

In the theory of abelian categories, a model example is the category of modules over a ring (for instance, the group ring of a group, in which case we have a category of representations). The morphisms are module homomorphisms (or intertwining operators in the case of representations). The symplectic analogue of this example is the “symplectic category” of representations of a Poisson manifold, in which objects are symplectic realizations and spaces of morphisms are the classical intertwiner spaces.

To avoid smoothness issues, we will be more restrictive and define the **representation category** of a Poisson manifold P to be the symplectic category in which the objects are symplectic P -torsors $S \rightarrow M$ which are (P, M) -Morita equivalences, and the morphism spaces are classical intertwiner spaces, $\text{Hom}(S_1, S_2) := \overline{S_2} *_P S_1$.

Composition relations are given by

$$L_{S_1 S_2 S_3} := \{([(z, x)], [(y, x)], [(z, y)])\} \subset \overline{S_3} *_P S_1 \times S_2 *_P \overline{S_1} \times S_3 *_P \overline{S_2} \quad (5.1)$$

where $[(a, b)] \in \overline{S'} *_P S$ denotes the image of $(a, b) \in \overline{S'} \times_P S$ under the natural projection.

Exercise

Check that the composition relation (5.1) is a lagrangian submanifold. (Hint: first prove it when P is a point, then use coisotropic reduction for the general case.)

Note that $\text{Hom}(S, S) = \overline{S} *_P S$ is symplectomorphic to the symplectic groupoid $\mathcal{G}(M)$, where $M = S/\mathcal{G}(P)$.

Exercise

Show that the composition relation in $\text{Hom}(S, S) = \mathcal{G}(M)$, where $M = S/\mathcal{G}(P)$, is just the graph of the groupoid multiplication $\mathcal{G}(M) \leftarrow \mathcal{G}(M) \times_{(s,t)} \mathcal{G}(M)$ in $\mathcal{G}(M) \times \overline{\mathcal{G}(M)} \times \mathcal{G}(M)$.

Finally, we define “invertible elements” in $\text{Hom}(S_1, S_2) = \overline{S_2} *_P S_1$ to be those lagrangian submanifolds which are the reductions of graphs of isomorphisms of symplectic realizations $S_1 \rightarrow S_2$ via the coisotropic submanifold $\overline{S_2} \times_P S_1$ of $\overline{S_2} \times S_1$. In particular, two symplectic realizations are “isomorphic” in the representation category of P if and only if they are isomorphic in the usual sense.

Proposition 5.1 *Two Poisson manifolds are Morita equivalent if and only if they have equivalent representation categories.*

PROOF:

Suppose that P_1 and P_2 are Morita equivalent, and let X be a (P_1, P_2) -Morita bimodule. Let $\mathcal{S}(P_i)$ denote the representation category of P_i , $i = 1, 2$. Then X induces an equivalence of symplectic categories $\mathcal{S}(P_2) \rightarrow \mathcal{S}(P_1)$: at the level of objects, a Morita bimodule $P_2 \leftarrow S \rightarrow M$ is mapped to the Morita bimodule $P_1 \leftarrow X *_P S \rightarrow M$; if $P_1 \leftarrow S' \rightarrow M'$ is an object in $\mathcal{S}(P_1)$, then $X *_P \overline{S'}$ is an object in $\mathcal{S}(P_2)$ such that S' and $X *_P \overline{X} *_P S'$ are isomorphic; at the level of morphisms, because $\overline{S_2} *_P S_1 \cong \overline{S_2} *_P \overline{X} *_P X *_P S_1$, we have a natural symplectomorphism

$$\text{Hom}(S_1, S_2) \cong \text{Hom}(X *_P S_1, X *_P S_2).$$

Conversely, suppose that $F : \mathcal{S}(P_2) \rightarrow \mathcal{S}(P_1)$ is an equivalence of symplectic categories, and let $P_2 \leftarrow S \rightarrow M$ be an object in $\mathcal{S}(P_2)$. Then there is a symplectomorphism from

$\text{Hom}(S, S) = \mathcal{G}(M)$ to $\text{Hom}(F(S), F(S)) = \mathcal{G}(M')$, where $M' = F(S)/\mathcal{G}(P_1)$. Since this symplectomorphism preserves the composition relation, it is a symplectic groupoid isomorphism. In particular, M and M' are isomorphic as Poisson manifolds, which implies that $F(S)$ is a (P_1, M) -Morita bimodule. If we take $S = \mathcal{G}(P_2)$, then $M = P_2$ and $F(S)$ is a (P_1, P_2) -Morita bimodule. \square

The equivalence in Remark 4.37 does not preserve intertwiner spaces (their symplectic structures are related by a factor of 2), so it does not contradict the result above.

References

- [1] ALEKSEEV, A.: *On Poisson actions of compact Lie groups on symplectic manifolds*. J. Diff. Geom. **45** (1997), 241–256.
- [2] ALEKSEEV, A., KOSMANN-SCHWARZBACH, Y., MEINRENKEN, E.: *Quasi-Poisson manifolds*. Canadian J. Math. **54** (2002), 3–29.
- [3] ALEKSEEV, A., MALKIN, A., MEINRENKEN, E.: *Lie group valued moment maps*. J. Diff. Geom. **48** (1998), 445–495.
- [4] BASS, H.: *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.
- [5] BAYEN, F., FLATO, M., FRØNSDAL, C., LICHNEROWICZ, A., STERNHEIMER, D.: *Deformation Theory and Quantization*. Ann. Phys. **111** (1978), 61–151.
- [6] BEER, W.: *On Morita equivalence of nuclear C^* -algebras*. J. Pure Appl. Algebra **26** (1982), 249–267.
- [7] BEHREND, K., XU, P.: *Differentiable stacks and gerbes*. In preparation.
- [8] BEHREND, K., XU, P., ZHANG, B.: *Equivariant gerbes over compact simple Lie groups*. C.R. Acad. Sci. Paris **336** (2003), 251–256.
- [9] BÉNABOU, J.: *Introduction to bicategories*. In: *Reports of the Midwest Category Seminar*, 1–77. Springer, Berlin, 1967.
- [10] BEREZIN, F.A.: *Some remarks about the associated envelope of a Lie algebra*, Funct. Anal. Appl. **1** (1967), 91–102.
- [11] BLECHER, D.: *On Morita’s fundamental theorem for C^* -algebras*. Math. Scand. **88** (2001), 137–153.
- [12] BROWN, L. G., GREEN, P., RIEFFEL, M.: *Stable isomorphism and strong Morita equivalence of C^* -algebras*. Pacific J. Math **71** (1977), 349–363.
- [13] BURSZTYN, H.: *Semiclassical geometry of quantum line bundles and Morita equivalence of star products*. Int. Math. Res. Not. **16** (2002), 821–846.
- [14] BURSZTYN, H., CRAINIC, M.: *Dirac structures, moment maps and quasi-Poisson manifolds*. Progress in Mathematics, Festschrift in Honor of Alan Weinstein, Birkhäuser, to appear. math.DG/0310445.
- [15] BURSZTYN, H., CRAINIC, M., WEINSTEIN, A., ZHU, C.: *Integration of twisted Dirac brackets*. Duke Math. J., to appear. math.DG/0303180.
- [16] BURSZTYN, H., RADKO, O.: *Gauge equivalence of Dirac structures and symplectic groupoids*. Ann. Inst. Fourier (Grenoble) **53** (2003), 309–337.
- [17] BURSZTYN, H., WALDMANN, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds*. Comm. Math. Phys. **228** (2002), 103–121.
- [18] BURSZTYN, H., WALDMANN, S.: *Completely positive inner products and strong Morita equivalence*. math.QA/0309402.
- [19] BURSZTYN, H., WEINSTEIN, A.: *Picard groups in Poisson geometry*. Moscow Math. J., to appear. math.SG/0304048.
- [20] CANNAS DA SILVA, A., WEINSTEIN, A.: *Geometric models for noncommutative algebras*. American Mathematical Society, Providence, RI, 1999.
- [21] CATTANEO, A.: *On the integration of Poisson manifolds, Lie algebroids, and coisotropic submanifolds*. Lett. Math. Phys., to appear. math.SG/0308180.

- [22] CATTANEO, A., FELDER, G.: *Poisson sigma models and symplectic groupoids*. In: N.P.Landsman, M. Pflaum, and M. Schlichenmaier, eds., *Quantization of singular symplectic quotients*, Progr. Math. 198, 61–93, Birkhäuser, Basel, 2001.
- [23] CATTANEO, A.S., INDELICATO, D.: *Formality and Star Products*, to appear in Proceedings of the Euroschool PQR2003.
- [24] CATTANEO, A., XU, P.: *Integration of twisted Poisson structures*. J. Geom. Phys., to appear. math.SG/0302268.
- [25] CONN, J.: *Normal forms for smooth Poisson structures*. Ann. of Math. **121** (1995), 565–593.
- [26] COSTE, A., DAZORD, P., WEINSTEIN, A.: *Groupeïdes symplectiques*. In: *Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2*, i–ii, 1–62. Univ. Claude-Bernard, Lyon, 1987. Scanned version available at math.berkeley.edu/~alanw/cdw.pdf.
- [27] COURANT, T.: *Dirac manifolds*. Trans. Amer. Math. Soc. **319** (1990), 631–661.
- [28] CRAINIC, M.: *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes*. Comment. Math. Helv. **78** (2003), 681–721 .
- [29] CRAINIC, M., FERNANDES, R.L.: *Integrability of Lie brackets*. Ann. of Math. **157** (2003), 575–620.
- [30] CRAINIC, M., FERNANDES, R.L.: *Integrability of Poisson brackets*. J. Diff. Geom., to appear. math.DG/0210152.
- [31] CRAINIC, M., FERNANDES, R.L.: *Linearization of Poisson structures*. Talks given at “PQR2003”, Brussels, and “AlanFest”, Vienna, 2003.
- [32] DAVIS, B.L., WADE, A. : *Vorobjev linearization of Poisson manifolds*, in preparation.
- [33] DAZORD, P.: *Groupeïdes symplectiques et troisième théorème de Lie “non linéaire”*. Lect. Notes in Math. **1416**, 39–74, Springer-Verlag, Berlin, 1990.
- [34] DUFOUR, J.-P., ZUNG, N.T.: *Poisson Structures and their Normal Forms*, book in preparation.
- [35] ELLIOT, G. A., LI, H.: *Morita equivalence of smooth noncommutative tori*, math.OA/0311502.
- [36] FEDOSOV, B.: *A simple geometrical construction of deformation quantization*, J. Diff. Geom. **40** (1994), 213–238.
- [37] FERNANDES, R.L., MONNIER, P. : *Linearization of Poisson brackets*. math.SG/0401273.
- [38] GERSTENHABER, M., SCHACK, S.D., : *Algebraic cohomology and deformation theory*. In: M. Hazewinkel and M. Gerstenhaber (eds.): *Deformation Theory of Algebras and Structures and Applications*, Kluwer Acad. Publ., Dordrecht, 1988, pp. 13 – 264.
- [39] GINZBURG, V. L., GOLUBEV, A.: *Holonomy on Poisson manifolds and the modular class*. Israel J. Math **122** (2001), 221–242.
- [40] GINZBURG, V. L., LU, J.-H.: *Poisson cohomology of Morita-equivalent Poisson manifolds*. Internat. Math. Res. Notices **10** (1992), 199–205.
- [41] GINZBURG, V., WEINSTEIN, A.: *Lie-Poisson structure on some Poisson Lie groups*. J. Amer. Math. Soc **5** (1992), 445–453.
- [42] GUALTIERI, M.: *Generalized complex geometry*. math.DG/0401221.
- [43] GURUPRASAD, K., HUEBSCHMANN, J., JEFFREY, L., WEINSTEIN, A.: *Group systems, groupoids, and moduli spaces of parabolic bundles* Duke Math. J. **89** (1997), 377–412.
- [44] GUTT, S.: *An explicit *-product on the cotangent bundle of a Lie group*, Lett. Math. Phys **7** (1983), 249–258.
- [45] HITCHIN, N. : *Generalized Calabi-Yau manifolds*. Q. J. Math. **54** (2003), 281–308.
- [46] JURCO, B., SCHUPP, P., WESS, J.: *Noncommutative line bundle and Morita equivalence*. Lett. Math. Phys. **61** (2002), 171–186.
- [47] KLIMČÍK, C., STROBL, T.: *WZW-Poisson manifolds*. J. Geom. Phys. **4** (2002), 341–344.
- [48] KONTSEVICH, M.: *Deformation quantization of Poisson manifolds, I*. Lett. Math. Phys. **66** (2003), to appear. Preprint q-alg/9709040.
- [49] KOSMANN-SCHWARZBACH, Y.: *Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory* . Progress in Mathematics, Festschrift in Honor of Alan Weinstein, Birkhäuser, to appear. math.SG/0310359.

- [50] LANCE, E. C.: *Hilbert C^* -modules. A toolkit for operator algebraists*. London Mathematical Society Lecture Note Series, 210. Cambridge University Press, Cambridge, 1995.
- [51] LANDSMAN, N. P.: *Mathematical Topics between Classical and Quantum Mechanics. Springer Monographs in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1998.
- [52] LANDSMAN, N. P.: *Quantized reduction as a tensor product* In: *Quantization of Singular Symplectic Quotients*, N.P. Landsman, M. Pflaum, M. Schlichenmaier, eds., 137–180, Birkhäuser, Basel, 2001.
- [53] LAURENT-GENGOUX, C., XU, P.: *Quantization of quasi-presymplectic groupoids and their hamiltonian spaces*. Progress in Mathematics, Festschrift in Honor of Alan Weinstein, Birkhäuser, to appear. math.SG/0311154.
- [54] LI, H.: *Strong Morita equivalence of higher-dimensional noncommutative tori*. J. Reine Angew. Math., to appear. math.OA/0303123.
- [55] LIE, S.: *Theorie der Transformationsgruppen*, (Zweiter Abschnitt, unter Mitwirkung von Prof. Dr. Friedrich Engel), Teubner, Leipzig, 1890.
- [56] LIU, Z.-J., WEINSTEIN, A., XU, P.: *Manin triples for Lie bialgebroids*. J. Diff. Geom. **45** (1997), 547–574.
- [57] MACKENZIE, K., XU, P.: *Integration of Lie bialgebroids*. Topology **39** (2000), 445–467.
- [58] MARSDEN, J., RATIU, T.: *Introduction to Mechanics and Symmetry*, Text in Applied Mathematics, Vol. 17, Springer-Verlag, 1994.
- [59] METZLER, D.: *Topological and smooth stacks*. math.DG/0306176.
- [60] MIKAMI, K., WEINSTEIN, A.: *Moments and reduction for symplectic groupoid actions*. Publ. RIMS, Kyoto Univ. **24** (1988), 121–140.
- [61] MOERDIJK, I., MRČUN, J.: *On integrability of infinitesimal actions*. Amer. J. Math. **124** (2002), 567–593.
- [62] MOERDIJK, I., MRČUN, J.: *Introduction to Foliations and Lie Groupoids*, Cambridge U. Press, Cambridge, 2003.
- [63] MOERDIJK, I., MRČUN, J.: *Lie groupoids, sheaves, and cohomology*, to appear in Proceedings of the Euroschool PQR2003.
- [64] MORITA, K.: *Duality for modules and its applications to the theory of rings with minimum condition*. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.
- [65] MOSER, J.: *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.
- [66] MRČUN, J.: *On isomorphisms of algebras of smooth functions*. Preprint math.DG/0309179.
- [67] NEST, R., TSYGAN, B.: *Algebraic index theorem*. Comm. Math. Phys. **172** (1995), 223–262.
- [68] PARK, J.-S., *Topological open p-branes*, Symplectic geometry and Mirror symmetry (Seoul 2000), 311–384, World Sci. Publishing, River Edge, NJ, 2001.
- [69] PRONK, D.: *Etendues and stacks as bicategories of fractions*. Compositio Math. **102** (1996), 243–303.
- [70] RADKO, O.: *A classification of topologically stable Poisson structures on a compact oriented surface*. J. Symp. Geometry. **1** (2002), 523–542.
- [71] RAEBURN, I., WILLIAMS, D.: *Morita equivalence and continuous-trace C^* -algebras*. Amer. Math. Soc., Providence, RI, 1998.
- [72] RIEFFEL, M. A.: *Induced representations of C^* -algebras*. Adv. Math. **13** (1974), 176–257.
- [73] RIEFFEL, M. A.: *Morita equivalence for C^* -algebras and W^* -algebras*. J. Pure Appl. Algebra **5** (1974), 51–96.
- [74] RIEFFEL, M. A.: *C^* -algebras associated with irrational rotations*. Pacific J. Math. **93** (1981), 415–429.
- [75] RIEFFEL, M. A.: *Deformation Quantization of Heisenberg Manifolds*. Commun. Math. Phys. **122** (1989), 531–562.
- [76] RIEFFEL, M. A., SCHWARZ, A.: *Morita equivalence of multidimensional noncommutative tori*. Internat. J. Math. **10** (1999), 289–299.
- [77] SCHWARZ, A.: *Morita equivalence and duality*. Nuclear Phys. B **534** (1998), 720–738.
- [78] ŠEVERA, P.: *Some title containing the words “homotopy” and “symplectic”, e.g. this one*. math.SG/0105080.

- [79] ŠEVERA, P., WEINSTEIN, A.: *Poisson geometry with a 3-form background*. Prog. Theo. Phys. Suppl. **144** (2001), 145–154.
- [80] SMALE, S.: *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.
- [81] TANG, X., WEINSTEIN, A.: *Quantization and Morita equivalence for constant Dirac structures on tori*. math.QA/0305413.
- [82] TSENG, H.-H., ZHU, C.: *Integrating Lie algebroids via stacks*. In preparation .
- [83] UCHINO, K.: *Lagrangian calculus on Dirac manifolds*. Preprint.
- [84] VAISMAN, I.: *Lectures on the geometry of Poisson manifolds*. Birkäuser, Basel, 1994.
- [85] VOROBJEV, Y.: *Coupling tensors and Poisson geometry near a single symplectic leaf*, in *Lie algebroids and related topics in differential geometry (Warsaw, 2000)*, Banach Center Publ., **54**, Polish Acad. Sci., Warsaw, 2001, 249–274.
- [86] WEINSTEIN, A.: *The symplectic “category”*. In: *Differential geometric methods in mathematical physics* (Clausthal, 1980), Springer, Berlin, 1982, 45–51.
- [87] WEINSTEIN, A.: *The local structure of Poisson manifolds*. J. Diff. Geom. **18** (1983), 523–557.
- [88] WEINSTEIN, A.: *Symplectic groupoids and Poisson manifolds*. Bull. Amer. Math. Soc. (N.S.) **16** (1987), 101–104.
- [89] WEINSTEIN, A.: *Coisotropic calculus and Poisson groupoids*. J. Math. Soc. Japan **40** (1988), 705–727.
- [90] WEINSTEIN, A.: *Lagrangian mechanics and groupoids*, Mechanics Day, W.F. Shadwick, P.S. Krishnaprasad, and T.S. Ratiu, eds., Fields Institute Communications, v. 7., Amer. Math. Soc., Providence, 1995, pp. 207–231.
- [91] WEINSTEIN, A.: *The modular automorphism group of a Poisson manifold*. J. Geom. Phys. **23** (1997), 379–394.
- [92] WEINSTEIN, A.: *Integrating the nonintegrable*. Preprint, 2004.
- [93] XU, P.: *Morita equivalence of Poisson manifolds*. Comm. Math. Phys. **142** (1991), 493–509.
- [94] XU, P.: *Morita equivalent symplectic groupoids*. In: *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, 291–311. Springer, New York, 1991.
- [95] XU, P.: *Classical intertwiner spaces and quantization*. Comm. Math. Phys. **164** (1994), 473–488.
- [96] XU, P.: *Momentum maps and Morita equivalence*. math.SG/0307319.